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Well-Posedness of Degenerate Integro-Differential Equations with Infinite Delay in Banach Spaces

By

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We are concerned with a class of degenerate integro-differential equations of second order in time in Banach spaces. We characterize their well-posedness using operator valued Fourier multipliers. These equations are important in several applied problems in physics and material science, especially for phenomena where memory effects are important. One such domain is viscoelasticity. We focus on the periodic case and we treat vector-valued Lebesgue, Besov and Trieblel-Lizorkin spaces. We note that in the Besov case, the results are applicable in particular to the scale of vector-valued Hölder spaces C^s, $0 < s < 1$. The definition of well posedness we adopt is a modification of the one used so far in the special cases. Thus, our results have as corollaries those obtained by several authors for first and second order integro-differential equations in the non-degenerate context.

In memory of my grandfather Manuel Antonio Aparicio Moreno

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LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers
$\mathbb{N}_0$	Set of nonnegative integers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\hat{f}(k)$	Fourier coefficient of f
$\tilde{a}$	Laplace transform of a
$\sum_\varphi$	Sector of angle φ
$\sigma(A)$	Spectrum of A
$\rho(A)$	Resolvent set of A
$R(\lambda, A)$	Resolvent of A
$\mathscr{L}(X, Y)$	Space of bounded linear operators from X to Y
$\mathscr{L}(X)$	Space of bounded linear operators on X
$\triangle = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$	Laplacian on a subset of $\mathbb{R}^n$
Δ^n	The n^{th} order difference operator
$W_0^{1,p}(\Omega)$, $W^{1,p}(\Omega)$	Classical Sobolev space of order on Ω
$L^p(\Omega)$	Space of Lebesgue p-integrable functions on Ω
$\mathscr{S}(\mathbb{R})$	The Schwartz space on $\mathbb{R}$
$\mathscr{D}(0, 2\pi)$	Space of all complex-valued infinitely differentiable functions on $[0, 2\pi]$
$\mathscr{D}'(0, 2\pi; X)$	X-valued distributions on $[0, 2\pi]$
$L^p(0, 2\pi; X)$	2π-Periodic Lebesgue-Bochner spaces
$B^s_{pq}(0, 2\pi, X)$	2π-Periodic Besov spaces
$F^s_{pq}(0, 2\pi, X)$	2π-Periodic Triebel-Lizorkin spaces
$C^s(0, 2\pi, X)$	2π-Periodic Hölder spaces

INTRODUCTION

In the present work, we consider the following problem which consists in a second order degenerate integro-differential equation with infinite delay in a complex Banach space X:

$$\begin{aligned}&(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds \\ &= \gamma_\infty u(t) + Au(t) + \textstyle\int_{-\infty}^{t} a(t-s)Au(s)ds \\ &\quad +b_\infty Bu(t) + \textstyle\int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \;\; 0 \le t \le 2\pi,\end{aligned} \tag{1}$$

and periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. We also consider the following variant of this problem, namely:

$$\begin{aligned}&(Mu)''(t) - (\Lambda u)'(t) - \frac{d}{dt}\int_{-\infty}^{t} c(t-s)u(s)ds \\ &= \gamma_\infty u(t) + Au(t) + \textstyle\int_{-\infty}^{t} a(t-s)Au(s)ds \\ &\quad +b_\infty Bu(t) + \textstyle\int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \;\; 0 \le t \le 2\pi,\end{aligned} \tag{2}$$

subject to the following three periodic boundary conditions $\Lambda u(0) = \Lambda u(2\pi)$, $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$. Here, A, B, Λ and M are closed linear operators in the Banach space X satisfying the assumption $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$, $a, b, c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0,\, 2\pi]$, and γ_∞, b_∞ are constants. We call the equation (1), equation of the type I and (2) equation of the type II. If $M = 0$ in the first problem, the second periodic boundary condition above becomes irrelevant. Observe that in the second problem: if $\Lambda = M = 0$, the three periodic boundary conditions disappear; if only $M = 0$, the three periodic boundary conditions are reduced to one; if only $\Lambda = 0$, the three periodic boundary conditions are become in two; and if $\Lambda \neq 0$ and $M \neq 0$ but $\Lambda = M$ or Λ is injective or M is injective, the three periodic boundary conditions also reduce to two.

Equations of the form (1) or (2) appear in a variety of applied problems. The case where the memory effect is absent has been studied by many authors. The monograph [25] by Favini and Yagi is devoted to degenerate problems. Evolutionary integro-differential equations arise typically in mathematical physics by constitutive laws pertaining to materials for which memory effects are important, when combined with the usual conservation laws such as balance of energy or balance of momentum. For details concerning the underlying physical principles, we refer to Coleman-Gurtin [17], Lunardi [36], Nunziato [37], and Prüss [40] (particularly Chapter II, Section 9) for work on the subject. Typical examples for $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are the functions $Ke^{-\omega t}t^{\mu}$ where $K \geq 0$, $\omega > 0$ and $\mu > -1$.

Several authors have considered particular cases of the above equation. Earlier papers: Lunardi [36], Da Prato-Lunardi [20], Nunziato [37], for example, use various techniques for the solvability of problems of this type. In the case of Hilbert spaces, the results obtained by these authors are complete. This is due to the fact that Plancherel's theorem is available in Hilbert space. When X is not a Hilbert space, this is no longer the case because of Kwapien's theorem which states that the validity of Plancherel's theorem for X-valued functions requires X to be isomorphic to Hilbert space (see for example Arendt-Bu [6]). Beginning with the papers by Weis [45, 46], Arendt-Bu [6], Arendt-Batty-Bu [4], it became possible to completely characterize well-posedness of the problem in periodic vector-valued function spaces. Initially, Arendt and Bu [6] dealt with the problem $u'(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$. Maximal regularity for the evolution problem in L^p was treated earlier by Weis [45, 46]. The study in the L^p context ($1 < p < \infty$) was made possible with the introduction of the concept of randomized boundedness (hereafter $R-$boundedness) to establish necessary conditions for operator-valued Fourier multipliers in this context. In addition in the L^p context, the space X must have the UMD property.

This was done initially by Weis [45, 46] for the evolutionary problem and then by Arendt-Bu [6] for periodic boundary conditions.

Our characterizations rely on the operator-valued Fourier multiplier theorems obtained by Arendt and Bu [6] on $L^p(0, 2\pi; X)$, Arendt and Bu [7] on $B^s_{pq}(0, 2\pi; X)$, and Bu and Kim [15] on $F^s_{pq}(0, 2\pi; X)$. We give characterizations of well-posedness of (1) and (2) in these spaces in terms of operator-valued Fourier multipliers and then we derive concrete conditions that allow us to apply these characterizations to integro-differential equations.

More recently, degenerate equations have attracted the attention of many authors. Both first and second order equations have been considered. The first order degenerate equation

$$(Mu)'(t) = Au(t) + f(t), \;\; 0 \le t \le 2\pi, \tag{3}$$

with periodic boundary condition $Mu(0) = Mu(2\pi)$, has been studied by Lizama and Ponce [35]. Under suitable assumptions on the modified resolvent operator associated to (3), they gave necessary and sufficient conditions to ensure the well-posedness of (3) in the Lebesgue-Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$.

Recently Bu [12] studied the following second order degenerate equation

$$(Mu')'(t) = Au(t) + f(t), \;\; 0 \le t \le 2\pi, \tag{4}$$

with periodic boundary conditions $u(0) = u(2\pi)$, $(Mu')(0) = (Mu')(2\pi)$. He also obtained necessary and sufficient conditions for the well-posedness of (4) in Lebesgue-Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$ under some suitable conditions on the modified resolvent operator associated to (4).

For references on degenerate equations and their relevance in concrete problems, we refer to the book [25] by Favini and Yagi. Other references are Barbu and Favini [8], Favaron and Favini [26]. For general evolutionary integro-differential equations, the reference [40] gives a very complete picture.

When more than one unbounded operators are involved in equations (1) or (2), a strengthening of the definition of well-posedness is necessary. The resulting definition (Definition 2.1.4 below) which we provide seems to be new in this context. In fact our definition is parallel to the usual one for partial differential equations, in the sense of Hadamard, namely existence, uniqueness and continuous dependence of the solution on the data of the problem. The definition given is consistent with the previously adopted ones in the case where only one unbounded operator appears in the equation.

We study equations (1) and (2) in the spaces of $2\pi-$periodic vector-valued functions: Lebesgue-Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi; X)$.

This work is organized as follows: In Chapter 1 we collect some preliminary results and definitions. In Chapter 2, we give necessary and sufficient conditions for well-posedness of the equation (1) in the Lebesgue Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin $F^s_{pq}(0, 2\pi; X)$ spaces in terms of operator-valued Fourier multipliers. Subsequently, we give concrete conditions on the data ensuring applicability of the results established. We stress that in the L^p case, the results require the space X to be UMD, and the concept of R-boundedness is essential. The latter first appeared in the context of evolution equations in the papers [45, 46] of Weis (see also the article [28]). In the other cases (namely $B^s_{pq}(0, 2\pi; X)$ and $F^s_{pq}(0, 2\pi; X)$), these restrictions are no longer needed but one requires instead higher order boundedness conditions on the " modified resolvents" involved. In Chapter 3, we study the equation (2) and obtain similar results as in

Chapter 2. We remark however that geometric conditions on the space X such as possessing nontrivial Fourier type bring simplifications to the conditions.

Recall that a Banach space X has nontrivial Fourier type if and only if it has nontrivial Radamacher type. These types are in general different. Such Banach spaces are also called B-convex (see [10], [33]). In the final Chapter 4, we consider some examples where above results apply. We single out the following problem as an example

$$\begin{cases} \dfrac{\partial}{\partial t}(m(x)\dfrac{\partial u(t,x)}{\partial t}) - \Delta\dfrac{\partial u(t,x)}{\partial t} \\ = \Delta u(t,x) + \int_{-\infty}^{t} a(t-s)\Delta u(s,x)ds + f(t,x),\ (t,x) \in [0,2\pi]\times\Omega, \\ u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0,\ (t,x)\in[0,2\pi]\times\partial\Omega, \\ u(0,x) = u(2\pi,x),\ m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t},\ x\in\Omega. \end{cases} \tag{5}$$

Here, Ω is an open subset of $\mathbb{R}^n$ and $\triangle$ is the Laplace operator.

We study the problem for periodic boundary conditions. Examples of this type were considered in Favini-Yagi [25, Example 6.1] as an evolutionary problem. They consider only the case when $a = 0$, that is, they do not incorporate the memory term in the equation. They restrict their study to the Hölder spaces. For periodic boundary conditions, we obtain complete characterization of well-posedness in the three scales of spaces: L^p, B^s_{pq}, and F^s_{pq}.

The above equation is a degenerate wave type equation with strong damping. If the operator $\triangle$ in the right hand side is replaced by $-\triangle$, then we have an equation of elliptic type. These have been much studied by A. Favini and his collaborators.

CHAPTER 1
PRELIMINARIES

In this chapter, we collect some results and definitions that will be used in the sequel.

1.1 Vector-valued Function Spaces

Let X be a complex Banach space. For $k \in \mathbb{Z}$, we define $e_k(t)$ by $e_k(t) = e^{ikt}$, $t \in \mathbb{R}$. For $x \in X$, we define the X-valued function $e_k \otimes x$ by $(e_k \otimes x)(t) = e^{ikt}x$, $t \in \mathbb{R}$.

We consider the following scale spaces:

1. **Lebesgue-Bochner spaces.** For $1 \leq p \leq \infty$, we denote $L^p(0, 2\pi; X)$ (denoted also $L^p_{2\pi}(\mathbb{R}; X)$, $1 \leq p \leq \infty$) of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0, 2\pi]$ is p-integrable, usual modification if $p = \infty$. The space is equipped with the norm

$$\|f\|_p = \|f\|_{L^p(0,2\pi,X)} = \begin{cases} \left(\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \|f(t)\|_X^p dt \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \\ \operatorname*{ess\,sup}_{t \in [0,2\pi]} \|f(t)\|_X & \text{if } p = \infty. \end{cases} \tag{1.1}$$

Let $1 \leq p < \infty$ and $m \in \mathbb{N}$. Then the 2π-periodic Sobolev spaces are defined by

$$W^{m,p}(0, 2\pi; X) = \{f \in L^p(0, 2\pi; X) : f^{(j)} \in L^p(0, 2\pi; X) \text{ for all } 0 \leq j \leq m\},$$

where $f^{(j)}$ is the jth distributional derivative of f. The space is equipped with the norm

$$\|f\|_{W^{m,p}(0,2\pi;X)} = \sum_{0\le j\le m} \|f^{(j)}\|_p.$$

2. **Besov spaces.** We briefly recall the the definition of 2π-periodic Besov spaces in the vector-valued case as presented in [7]. Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathscr{D}(0,2\pi)$ be the space of all infinitely differentiable functions on $[0,2\pi]$ equipped with the locally convex topology given by the family of seminorms

$$\|f\|_\alpha = \sup_{x\in[0,2\pi]} |f^{(\alpha)}(x)|$$

for $\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $\mathscr{D}'(0,2\pi;X) := \mathscr{L}(\mathscr{D}(0,2\pi),X)$ be the space of all bounded linear operators from $\mathscr{D}(0,2\pi)$ to X (X-valued distributions). In order to define the Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t\in\mathbb{R} : |t|\le 2\},\ I_k = \{t\in\mathbb{R} : 2^{k-1} < |t| \le 2^k\}$$

for $k\in\mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathscr{S}(\mathbb{R})$ satisfying $\mathrm{supp}(\phi_k) \subset \overline{I}_k$ for each $k\in\mathbb{N}_0$, $\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1$ for $x\in\mathbb{R}$, and for each $\alpha\in\mathbb{N}_0$, $\sup_{x\in\mathbb{R},k\in\mathbb{N}_0} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty$. For $f\in\mathscr{D}'((0,2\pi),X)$, we set $\hat{f}(k) = f(e_k)$. We call $\hat{f}(k)$ the kth Fourier coefficient of f.

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1\le p,q\le\infty$, $s\in\mathbb{R}$, the X-valued 2π-periodic Besov space is denoted by $B^s_{pq}((0,2\pi),X)$ and defined to be the set

$$\left\{ f\in\mathscr{D}'((0,2\pi),X) : \|f\|^s_{pq} := \left(\sum_{j\ge 0} 2^{sjq} \|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)\hat{f}(k)\|_p^q\right)^{1/q} < \infty \right\}$$

with the usual modification if $q=\infty$, namely

$$\|f\|^s_{p,\infty} = \sup_{j\ge 0} 2^{sj} \|\sum_{k\in\mathbb{Z}} e_k\otimes\phi_j(k)\hat{f}(k)\|_p.$$

The expression $\|f\|_{pq}^{s}$ is a norm. It is known that $B_{pq}^{s}(0,2\pi;X)$ is independent of the choice of ϕ, and different choices of ϕ lead to equivalent norms $\|\cdot\|_{pq}^{s}$. Equipped with the norm $\|\cdot\|_{pq}^{s}$, $B_{pq}^{s}(0,2\pi;X)$ is a Banach space.

It is also known that if $s_1 \leq s_2$, then $B_{pq}^{s_2}(0,2\pi;X) \subset B_{pq}^{s_1}(0,2\pi;X)$ and the embedding is continuous [7]. When $s>0$, it is proved in [7] that $B_{pq}^{s}(0,2\pi;X) \subset L^p(0,2\pi;X)$ and the embedding is continuous; moreover, $f \in B_{pq}^{s+1}(0,2\pi;X)$ if and only if f is differentiable a.e on $[0,2\pi]$ and $f' \in B_{pq}^{s}(0,2\pi;X)$. In the case where $p=q=\infty$ and $0<s<1$ we have that $B_{\infty,\infty}^{s}(0,2\pi;X)$ corresponds to the space $C^s(0,2\pi;X)$ of Hölder continuous functions with equivalent norm

$$\|f\|_{C^s(0,2\pi;X)} = \sup_{t_1\neq t_2} \frac{\|f(t_2)-f(t_1)\|_X}{|t_2-t_1|^s} + \|f\|_\infty .$$

3. **Triebel-Lizorkin spaces.** Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed with ϕ and $\phi(\mathbb{R})$ as above. For $1 \leq p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$, the X-valued 2π-periodic Triebel-Lizorkin space with parameters s, p and q is denoted by $F_{pq}^{s}(0,2\pi;X)$ and defined by the set

$$\left\{ f \in \mathscr{D}'((0,2\pi),X) : \|f\|_{pq}^{s} := \left\| \left(\sum_{j\geq 0} 2^{sjq} \| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\|_X^q \right)^{1/q} \right\|_p < \infty \right\}$$

with the usual modification if $q=\infty$, namely

$$\|f\|_{p,\infty}^{s} = \left\| \sup_{j\geq 0} 2^{sj} \| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\|_X \right\|_p .$$

The expression $\|f\|_{pq}^{s}$ is a norm. It is known that set $F_{pq}^{s}(0,2\pi;X)$ is independent of the choice of ϕ, and different choices of ϕ lead to equivalent norms $\|\cdot\|_{pq}^{s}$. Equipped with the norm $\|\cdot\|_{pq}^{s}$, $F_{pq}^{s}(0,2\pi;X)$ is a Banach space.

It is also known that if $s_1 \leq s_2$, then $F_{pq}^{s_2}(0,2\pi;X) \subset F_{pq}^{s_1}(0,2\pi;X)$ and the embedding is continuous [15]. When $s>0$, it is proved in [15] that $F_{pq}^{s}(0,2\pi;X) \subset L^p(0,2\pi;X)$ and the embedding is continuous; moreover, as in the case of $B_{pq}^{s}(0,2\pi;X)$,

$f \in F^{s+1}_{pq}(0, 2\pi; X)$ if and only if f is differentiable a.e on $[0, 2\pi]$ and $f' \in F^s_{pq}(0, 2\pi; X)$. The exceptional case $p = \infty$ will not be considered here. We refer to Schmeisser-Triebel [42, Section 3.4.2] for a discussion. Note that $F^s_{pp}(0, 2\pi; X) = B^s_{pp}(0, 2\pi; X)$, and this follows from the definition.

1.2 UMD spaces and Fourier Type

Let X be a Banach space and $1 < p < \infty$. For $f \in L^p(\mathbb{R}; X)$ and $0 < \varepsilon < R$, let

$$(H_{\varepsilon,R}f)(t) = \frac{1}{\pi}\int_{\varepsilon\le|t-s|\le R}\frac{f(s)}{t-s}ds = (\psi_{\varepsilon,R} * f)(t),\ t \in \mathbb{R},$$

where

$$\psi_{\varepsilon,R}(t) = \begin{cases} \dfrac{1}{\pi t} & \text{if } \varepsilon \le t \le R, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\psi_{\varepsilon,R} \in L^1(\mathbb{R})$, then $H_{\varepsilon,R} \in \mathscr{L}(L^p(\mathbb{R}; X))$ by Young's inequality. For $f \in \mathscr{S}(\mathbb{R})$, the Hilbert transform of f is given by:

$$Hf = \lim_{\substack{\varepsilon\downarrow 0 \\ R\to\infty}} H_{\varepsilon,R}f.$$

A Banach space X is said to be UMD, if the Hilbert transform extends to a bounded operator on $L^p(\mathbb{R}, X)$, for some (and then all) $1 < p < \infty$.

Another important notion in Banach space theory is that of Fourier type for a Banach space. Conditions for Fourier multipliers in $B^s_{pq}(0, 2\pi; X)$ are simplified when the Banach space involved has nontrivial Fourier type. The Hausdorff-Young inequality states that for $1 \le p \le 2$, the Fourier transform maps $L^p(\mathbb{R}) := L^p(\mathbb{R}; \mathbb{C})$ continuously into $L^{p'}(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{p'} = 1$, with the convention that $p' = \infty$ when $p = 1$. In particular, when $p = 2$, Plancherel's theorem holds. When X is a Banach space and one considers $L^p(\mathbb{R}; X)$, the situation is no longer the same. It is known that Plancherel's theorem holds if and only if X is a Hilbert space (see e.g. [2], [5], [6], [28]). For an arbitrary Banach space, the Hausdorff-Young theorem holds with $p = 1$ as a consequence of the Riemann-Lebesgue Lemma. A Banach space is said to have non-trivial Fourier type if the Hausdorff-Young theorem holds true for some $p \in (1, 2]$. By a result of Bourgain, UMD spaces are examples of spaces with nontrivial Fourier type (see [28], [4]). Superreflexive Banach spaces have nontrivial Fourier type ([10, Proposition 3]). However, there exist non reflexive Banach spaces with nontrivial Fourier type. For Banach spaces, having nontrivial Fourier type is equivalent to being B-convex (see [10], [33]). The implications of the property of having non trivial Fourier type are studied in Giradi-Weis [28]. The multiplier theorems are greatly simplified.

1.3 Operator-Valued Fourier Multipliers

Let X be a complex Banach space. For a function $f \in L^1(0, 2\pi; X)$, we denote by $\hat{f}(k), k \in \mathbb{Z}$ its kth Fourier coefficient, namely:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)dt,$$

where $e_k(t) = e^{ikt}$, $t \in \mathbb{R}$.

The following observation is important for the problems we study in Chapter 2 and Chapter 3.

Let $u \in L^1(0, 2\pi; X)$. We denote again by u its periodic extension to $\mathbb{R}$. Let $a \in L^1(\mathbb{R}_+)$. We consider the the function

$$F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds,\ t \in \mathbb{R}.$$

Since

$$F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds = \int_0^{\infty} a(s)u(t-s)ds, \tag{1.2}$$

then $\|F\|_1 \leq \|a\|_{L^1(\mathbb{R}_+)}\|u\|_1$ and F is periodic of period $T = 2\pi$, as u. Now using Fubini's theorem and (1.2) we obtain, for $k \in \mathbb{Z}$, that

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k), k \in \mathbb{Z} \tag{1.3}$$

where $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t) dt$ denotes the Laplace transform of a. This identity plays a crucial role in the sequel when we consider integro-differential equations.

Let X, Y be Banach spaces. We denote by $\mathscr{L}(X, Y)$ the set of all bounded linear operators from X to Y. When $X = Y$, we write simply $\mathscr{L}(X)$.

For results about operator-valued Fourier multipliers and R-boundedness (used in the next Chapter), we refer to Amann [2], Bourgain [9], Clément-de Pagter-Sukochev-Witvliet [19], Weis [45, 46], Girardi-Weis [27], [28], Kunstmann-Weis [33], Clément- Prüss [18] and Arendt-Bu [6]. The scalar case is presented for example in Schmeisser-Triebel [42, Chapter 3]. This reference also considers the case where X is a Hilbert space (Chapter 6). Here, we will merely present the corresponding definitions.

We give the definition of operator-valued Fourier multipliers in each of the cases that will be of interest to us (See [6, 7, 15]). First, in the case of Lebesgue spaces, we have:

Definition 1.3.1. *Let X and Y be Banach spaces. For $1 \leq p \leq \infty$, we say that a sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathscr{L}(X, Y)$ is an L^p-Fourier multiplier, if for each $f \in L^p(0, 2\pi; X)$ there exists $u \in L^p(0, 2\pi; Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

In the case of Besov spaces, we have the following.

Definition 1.3.2. *Let X and Y be Banach spaces and let $(M_k)_{k \in \mathbb{Z}} \subset \mathscr{L}(X, Y)$ and $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$. We say that $(M_k)_{k \in \mathbb{Z}}$ is a B^s_{pq}-Fourier multiplier, if for each $f \in B^s_{pq}(0, 2\pi; X)$ there exists $u \in B^s_{pq}(0, 2\pi; Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$

Finally, in the case of Triebel-Lizorkin spaces, we have the following

Definition 1.3.3. *Let X and Y be Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ and $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. We say that $(M_k)_{k\in\mathbb{Z}}$ is an F^s_{pq}-Fourier multiplier, if for each $f \in F^s_{pq}(0,2\pi;X)$ there exists $u \in F^s_{pq}(0,2\pi;Y)$ such that*

$$\hat{u}(k) = M_k \hat{f}(k) \textit{ for all } k \in \mathbb{Z}.$$

From the uniqueness theorem of Fourier series, it follows that u is uniquely determined by f in each of the above cases.

We denote by $\mathscr{Y} = \mathscr{Y}(X)$ any of the following spaces of X-valued functions: $L^p(0,2\pi;X)$, $1 \le p \le \infty$; $B^s_{pq}(0,2\pi;X)$, $1 \le p,q \le \infty$, $s > 0$; $F^s_{pq}(0,2\pi;X)$, $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$. We define the following set

$$\mathscr{Y}^{[1]} = \{u \in \mathscr{Y} : u \text{ is almost everywhere differentiable and } u' \in \mathscr{Y}\}$$

and

$$\mathscr{Y}^{[1]}_{per} = \{u \in \mathscr{Y} : \exists v \in \mathscr{Y}, \text{ such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}$$

In the case that $\mathscr{Y} = L^p(0,2\pi;X)$, $\mathscr{Y}^{[1]}$ is denoted by $W^{1,p}(0,2\pi;X)$ and $\mathscr{Y}^{[1]}_{per}$ by $W^{1,p}_{per}(0,2\pi;X)$. In the case that $\mathscr{Y} = B^s_{pq}(0,2\pi;X)$, $\mathscr{Y}^{[1]} = B^{s+1}_{pq}(0,2\pi;X)$. In the case that $\mathscr{Y} = F^s_{pq}(0,2\pi;X)$, $\mathscr{Y}^{[1]} = F^{s+1}_{pq}(0,2\pi;X)$.

Remark 1.3.4.

Using integration by parts, the fact that $\mathscr{Y} \subset L^1(0,2\pi,X)$ and the uniqueness theorem of Fourier coefficients, we have that

$$\begin{aligned} \mathscr{Y}^{[1]}_{per} &= \{u \in \mathscr{Y}^{[1]} : u(0) = u(2\pi)\}, \\ \mathscr{Y}^{[1]}_{per} &= \{u \in \mathscr{Y}^{[1]} : \widehat{u'}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}. \end{aligned} \tag{1.4}$$

Therefore, if $u \in \mathscr{Y}^{[1]}_{per}$, then u has a unique continuous representative such that $u(0) = u(2\pi)$. We always identify u with this continuous extension.

Remark 1.3.5.

It is clear from the definitions that:

(a) If $(M_k)_{k\in\mathbb{Z}}, (N_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ are $\mathscr{Y}$-Fourier multipliers and α, β are constants, then $(\alpha M_k + \beta N_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ is a $\mathscr{Y}$-Fourier multiplier as well.

(b) If $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ and $(N_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(Y,Z)$ are $\mathscr{Y}$-Fourier multipliers, then $(N_k M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Z)$ is a $\mathscr{Y}$-Fourier multiplier as well. In particular, when $X = Y = Z$, if $(M_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers, then $(N_k M_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier. So in this case, the set of $\mathscr{Y}$-Fourier multipliers is an algebra.

We use the following results:

Proposition 1.3.6. *[6, Fejer's Theorem] Let $f \in L^p(0,2\pi;X)$, then one has*

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes \hat{f}(k)$$

in $L^p(0,2\pi;Y)$.

The following lemma is a version of Hille's theorem.

Lemma 1.3.7. *[6, Lemma 3.1] Let A be a closed operator on a Banach space X and $f, g \in L^p(0,2\pi;X)$, where $1 \le p < \infty$. The following assertions are equivalent.*

1. *$f(t) \in D(A)$ and $Af(t) = g(t)$ a.e. $t \in [0,2\pi]$;*
2. *$\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$.*

The following proposition characterizes Fourier multipliers.

Proposition 1.3.8. *[6, Proposition 1.1] Let $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ be a sequence. Then, $(M_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier if and only if there exists*

$$M \in \mathscr{L}(L^p(0,2\pi;X), L^p(0,2\pi;Y))$$

such that $\widehat{Mf}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$ and all $f \in L^p(0,2\pi;X)$. We call M the operator associated with $(M_k)_{k\in\mathbb{Z}}$. One has

$$f = \lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \otimes M_k \hat{f}(k)$$

in $L^p(0,2\pi;Y)$ for all $f \in L^p(0,2\pi;X)$.

Similar results hold in $B^s_{pq}(0,2\pi;X)$ and $F^s_{pq}(0,2\pi;X)$.

For the regularity of solutions, the following lemma is important.

Lemma 1.3.9. *[6, Lemma 2.2] Let $1 \leq p < \infty$. Let $M_k \in \mathscr{L}(X)$, $k \in \mathbb{Z}$. The following assertions are equivalent:*

1. *$(M_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier such that the associated operator $M \in \mathscr{L}(L^p(0, 2\pi; X))$ maps $L^p(0, 2\pi; X)$ into $W^{1,p}_{per}(0, 2\pi; X)$;*
2. *$(kM_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier.*

Again, analogous results are valid in the scales B^s_pq and F^s_pq.

In order to give the characterization of operator-valued Fourier multipliers, we need some more definitions.

Let $\{a_k : k \in \mathbb{Z}\} \subset \mathbb{C}$ be a scalar sequence, we denote by $\Delta a_k = a_{k+1} - a_k$ the first order difference. It is obvious that Δ is linear: $\Delta(a_k + b_k) = \Delta a_k + \Delta b_k$; $\Delta(\lambda a_k) = \lambda \Delta a_k$. Another property used frequently is $\Delta(a_k b_k) = a_k \Delta b_k + (\Delta a_k) b_k$. Define Δ^n inductively by: $\Delta^{n+1} a_k = \Delta \Delta^n a_k$ for all $n \in \mathbb{N}$. Then Δ^n is given by

$$\Delta^n a_k = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} a_{k+j}.$$

We also use this notation for sequences of operators.

The Marcinkiewicz conditions on the sequences $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X, Y)$ were used in [4], [6], [7], [13], [15] to study Fourier multipliers for vector-valued spaces. The Marcinkiewicz condition of order one is:

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\|) < \infty. \tag{1.5}$$

The Marcinkiewicz condition of order two is:

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\|) < \infty. \tag{1.6}$$

Finally, the Marcinkiewicz condition of order three is:

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\| + |k|^3\|\Delta^3 M_k\|) < \infty. \tag{1.7}$$

We state the operator-valued Fourier multiplier theorems for Besov and Triebel-Lizorkin spaces, for L^p spaces we need the concept of R-boundedness discussed in the next section.

Theorem 1.3.10. *[7, Theorem 4.5]*

1. *Let X and Y be arbitrary Banach spaces and let $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ be a sequence satisfying the Marcinkiewicz condition of order two. Then for $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $B^s_{pq}(0, 2\pi; X)$ to $B^s_{pq}(0, 2\pi; Y)$.*
2. *Let X and Y Banach spaces having nontrivial Fourier type and let $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ be a sequence satisfying the Marcinkiewicz condition of order one. Then for $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $B^s_{pq}(0, 2\pi; X)$ to $B^s_{pq}(0, 2\pi; Y)$.*

For Triebel-Lizorkin spaces, we have the following.

Theorem 1.3.11. *[15, Theorem 3.2]*
If $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ satisfies the Marcinkiewicz condition of order 2, then $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $F^s_{pq}(0, 2\pi; X)$ to $F^s_{pq}(0, 2\pi; Y)$ for $1 < q < \infty$, $1 < r \le \infty$ and $s \in \mathbb{R}$.
If $(M_k)_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order 3, then $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier from $F^s_{pq}(0, 2\pi; X)$ to $F^s_{pq}(0, 2\pi; Y)$ for $1 \le q \le \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$.

Remark 1.3.12.

(a) If $(kM_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier, then $(M_k)_{k\in\mathbb{Z}}$ is also a $\mathscr{Y}$-Fourier multiplier.

(b) If $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X,Y)$ is a $\mathscr{Y}$-Fourier multiplier, then there exists a bounded linear operator $T \in \mathscr{L}(\mathscr{Y}(X), \mathscr{Y}(Y))$ satisfying $\widehat{(Tf)}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$. This implies in particular that the sequence $(M_k)_{k\in\mathbb{Z}}$ must be bounded.

1.4 Rademacher Boundedness and L^p-multipliers

For $j \in \mathbb{N}$, denote by r_j the j-th Rademacher function on $[0, 1]$, i.e. $r_j(t) = sgn(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

The important concept of R-boundedness for a family of bounded linear operators is defined as follows:

Definition 1.4.1. *A family $\mathbf{T} \subset \mathscr{L}(X,Y)$ is called R-bounded if there exists $c_q \geq 0$ such that*

$$\|\sum_{j=1}^{n} r_j \otimes T_j x_j\|_{L^q(0,1;X)} \leq c_q \|\sum_{j=1}^{n} r_j \otimes x_j\|_{L^q(0,1;X)} \tag{1.8}$$

for all $T_1, ..., T_n \in \mathbf{T}$, $x_1, ..., x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. We denote by $R_q(\mathbf{T})$ the smallest constant c_q such that (1.8) holds.

Now we state the Marcinkiewicz operator-valued multiplier theorem.

Theorem 1.4.2. *[6, Theorem 1.3] Let X, Y be a UMD-spaces. Let $M_k \in \mathscr{L}(X,Y)$, for all $k \in \mathbb{Z}$. If the set $\{k(M_{k+1} - M_k) : k \in \mathbb{Z}\}$ and $\{M_k : k \in \mathbb{Z}\}$ are R-bounded, then $(M_k)_{k\in\mathbb{Z}}$ is an L^p-Fourier multiplier for $1 < p < \infty$.*

We also used the following results.

Proposition 1.4.3. *[6, Proposition 1.1] Let X be a Banach space and let $(M_k)_{k\in\mathbb{Z}}$ be an L^p-Fourier multiplier for $1 \leq p < \infty$. Then the set $\{M_k : k \in \mathbb{Z}\}$ is R-bounded.*

We now state the important Kahane Contraction Principle

Lemma 1.4.4. *[6, Lemma 1.7] We have for a Banach space X:*

$$||\sum_{j=1}^{n} r_j \otimes \lambda_j x_j||_{L^q(0,1;X)} \leq 2 \max_{j=1,\cdots,m} |\lambda_j| ||\sum_{j=1}^{n} r_j \otimes x_j||_{L^q(0,1;X)} \tag{1.9}$$

for all $\lambda_1, \cdots, \lambda_m \in \mathbb{C}$ and $x_1, \cdots, x_n \in X$, where $1 \leq q < \infty$.

Note that the factor 2 in the right hand side is not needed if we dealing with real scalars.

Remark 1.4.5.

Several useful properties of R-bounded families can be found in the monograph of Denk-Hieber-Prüss [23, Section 3], see also [6, 19, 22, 33]. We collect some of them here.

a) Any finite subset of $\mathscr{L}(X)$ is is R-bounded.

b) If $\mathbf{S} \subset \mathbf{T} \subset \mathscr{L}(X)$ and $\mathbf{T}$ is R-bounded, then $\mathbf{S}$ is R-bounded and $R_p(\mathbf{S}) \leq R_p(\mathbf{T})$.

c) Let $\mathbf{S}, \mathbf{T} \subset \mathscr{L}(X)$ be R-bounded sets. Then $\mathbf{S} \cdot \mathbf{T} := \{S \cdot T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R-bounded and

$$R_p(\mathbf{S} \cdot \mathbf{T}) \leq R_p(\mathbf{S}) \cdot R_p(\mathbf{T}).$$

d) Let $\mathbf{S}, \mathbf{T} \subset \mathscr{L}(X)$ be R-bounded sets. Then $\mathbf{S}+\mathbf{T} := \{S+T : S \in \mathbf{S}, T \in \mathbf{T}\}$ is R- bounded and

$$R_p(\mathbf{S} + \mathbf{T}) \leq R_p(\mathbf{S}) + R_p(\mathbf{T}).$$

e) If $\mathbf{T} \subset \mathscr{L}(X)$ is R- bounded, then $\mathbf{T} \cup \{0\}$ is R-bounded and $R_p(\mathbf{T} \cup \{0\}) = R_p(\mathbf{T})$.

f) If $\mathbf{S}, \mathbf{T} \subset \mathscr{L}(X)$ are R- bounded, then $\mathbf{T} \cup \mathbf{S}$ is R-bounded and

$$R_p(\mathbf{T} \cup \mathbf{S}) \leq R_p(\mathbf{S}) + R_p(\mathbf{T}).$$

g) Also, each subset $M \subset \mathscr{L}(X)$ of the form $M = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded (I denotes the identity operator on X).

The proofs of a), e), f), and g) rely on Kahane's contraction principle.

We sketch a proof of f) : Since we assume that $\mathbf{S}, \mathbf{T} \subset \mathscr{L}(X)$ are R-bounded, it follows from e) (which is a consequence of Kahane's contraction principle) that $\mathbf{S} \cup \{0\}$ and $\mathbf{T} \cup \{0\}$ are $R-$bounded. We now observe that $\mathbf{S} \cup \mathbf{T} \subset \mathbf{S} \cup \{0\} + \mathbf{T} \cup \{0\}$. Then using d) and b) we conclude that $\mathbf{S} \cup \mathbf{T}$ is $R-$bounded.

Remark 1.4.6.

If $X = Y$ is a UMD space and $M_k = m_k I$ with $m_k \in \mathbb{C}$, then the Marcinkiewicz condition

$$\sup_k |m_k| + \sup_k |k(m_{k+1} - m_k)| < \infty$$

implies that the set $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier. (see [6] or [2, Theorem 4.4.3]). This is the vector-valued Marcinkiewicz multiplier theorem.

CHAPTER 2
WELL-POSEDNESS OF THE EQUATION OF TYPE I

We give necessary and sufficient conditions for well-posedness of the equation of type I in the Lebesgue Bochner spaces $L^p(0, 2\pi; X)$, Besov spaces $B^s_{pq}(0, 2\pi; X)$ and Triebel-Lizorkin $F^s_{pq}(0, 2\pi; X)$ spaces in terms of operator-valued Fourier multipliers. After that, we give concrete conditions on the data ensuring applicability of the results established before. We stress that in the L^p case, the results require the space X to be UMD (this is equivalent to the continuity of the Hilbert transform on $L^p(\mathbb{R}, X)$, $1 < p < \infty$) and the concept of R-boundedness. The latter first appeared in the context of evolution equations in the papers [45, 46] of L. Weis (see also the article [28]). In the other cases (namely $B^s_{pq}(0, 2\pi; X)$ and $F^s_{pq}(0, 2\pi; X)$), these restrictions are no longer needed but one requires instead higher order boundedness conditions on the modified "resolvents" involved.

2.1 The General Well-Posedness Result.

In this section, we establish the general maximal regularity result for solutions of the problem

$$(TIP_1^2)\begin{cases} (Mu')'(t) - \Lambda u'(t) - \dfrac{d}{dt}\displaystyle\int_{-\infty}^{t} c(t-s)u(s)ds \\ = \gamma_\infty u(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds \\ \quad +b_\infty Bu(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \quad 0 \le t \le 2\pi, \\ u(0) = u(2\pi) \text{ and } (Mu)'(0) = (Mu)'(2\pi) \end{cases}$$

in the vector-valued Lebesgue, Besov, and Triebel-Lizorkin spaces. Here A, B, Λ and M are closed linear operators in a Banach space X satisfying $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$, $a, b, c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0,\, 2\pi]$, and γ_∞, b_∞ are constants. The results are in terms of operator-valued Fourier multipliers.

Let a, b, c be complex valued functions and γ_∞, b_∞ be constants. We define the (M, Λ)-resolvent set of A and B by

$$\begin{aligned}
&\rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B) := \{\lambda \in \mathbb{C} : \\
&\qquad \lambda^2 M - (1+\tilde{a}(\lambda))A - (b_\infty + \tilde{b}(\lambda))B - \lambda\Lambda\lambda\tilde{c}(\lambda)I - \gamma_\infty I : \\
&\qquad\qquad D(A) \cap D(B) \to X \text{ is bijective and} \\
&\quad [\lambda^2 M - (1+\tilde{a}(\lambda))A - (b_\infty + \tilde{b}(\lambda))B - \lambda\Lambda - \lambda\tilde{c}(\lambda)I - \gamma_\infty I]^{-1} \in \mathscr{L}(X)\}
\end{aligned}$$

$\lambda \in \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ if and only if the operator

$$[\lambda^2 M - (1+\tilde{a}(\lambda))A - (b_\infty + \tilde{b}(\lambda))B - \lambda\Lambda - \lambda\tilde{c}(\lambda)I - \gamma_\infty I]^{-1}$$

is a linear continuous isomorphism from X to $D(A) \cap D(B)$. Here we consider $D(A)$, $D(B)$, $D(\Lambda)$ and $D(M)$ as normed spaces equipped with their respective graph norms, e.g. the graph norm of $D(A)$ is given by $\|x\|_{D(A)} = \sqrt{\|x\|_X^2 + \|Ax\|_X^2}$ or and equivalent form. These are Banach spaces since all the operators are closed.

For $a \in L^1(\mathbb{R}_+)$, $u \in \mathscr{Y}$, we denote by $a * u$ the function

$$(a * u)(t) := \int_{-\infty}^{t} a(t-s)u(s)ds. \tag{2.1}$$

Since $\mathscr{Y} \subset L^1(0, 2\pi; X)$, then $a * u \in L^1(0, 2\pi; X)$ and $(a*u)(0) = (a*u)(2\pi)$ by (1.2). With this notation we may rewrite (1) in the following way

$$\begin{aligned}
&(Mu')'(t) - \Lambda u'(t) - \tfrac{d}{dt}(c * u)(t) \\
&= \gamma_\infty u(t) + Au(t) + (a * Au)(t) + b_\infty Bu(t) + (b * Bu)(t) + f(t),\ 0 \le t \le 2\pi.
\end{aligned}$$

If a, b, $c \in L^1(\mathbb{R}_+)$ and $u \in L^1(0,2\pi; D(A)) \cap L^1(0,2\pi; D(B))$, then $c*u$, $a*Au$, $a*Bu \in L^1(0,2\pi;X)$ by (1.2) and $\widehat{(c*u)}(k) = \tilde{c}(ik)\hat{u}(k)$, $\widehat{(a*Au)}(k) = \tilde{a}(ik)A\hat{u}(k)$ and $\widehat{(b*Bu)}(k) = \tilde{b}(ik)B\hat{u}(k)$ by (1.3). If additionally we have that $\frac{d}{dt}(c*u) \in L^1(0,2\pi;X)$, then $c*u \in W^{1,1}(0,2\pi;X)$ and $(c*u)(0) = (c*u)(2\pi)$. In this case, $\widehat{\frac{d}{dt}(c*u)}(k) = ik\tilde{c}(ik)\hat{u}(k)$ by (1.4).

In what follows, we adopt the following notation:

$$a_k := \tilde{a}(ik), b_k := \tilde{b}(ik), c_k := \tilde{c}(ik). \tag{2.2}$$

Remark 2.1.1.

The sequences $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$ and, $(c_k)_{k\in\mathbb{Z}}$ so defined are bounded. In fact by the Riemann-Lebesgue lemma, $\lim_{|k|\to\infty}(a_k) = 0$, and similarly for $(b_k)_{k\in\mathbb{Z}}$ and $(c_k)_{k\in\mathbb{Z}}$. Moreover, $(a_kI)_{k\in\mathbb{Z}}$, $(b_kI)_{k\in\mathbb{Z}}$ and $(c_kI)_{k\in\mathbb{Z}}$ define $\mathscr{Y}$-Fourier multipliers.

We now give the definition of solutions of (TIP_1^2) in the cases of interest to us.

Definition 2.1.2. *A function $u \in \mathscr{Y}$ is called a strong $\mathscr{Y}$-solution of (TIP_1^2) if $u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B)) \cap \mathscr{Y}_{per}^{[1]}$, $u' \in \mathscr{Y}(D(\Lambda)) \cap \mathscr{Y}(D(M))$, $Mu' \in \mathscr{Y}_{per}^{[1]}$, and equation (1) holds for almost all $t \in [0,2\pi]$.*

We have the following:

Lemma 2.1.3. *Let X be a Banach space, let A, B, Λ, M be closed linear operators in X such that $D(A)\cap D(B) \subset D(\Lambda)\cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ constants and a_k, b_k, c_k are defined by (2.2) and u is a strong $\mathscr{Y}$-solution of (TIP_1^2). Then*

$$[-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]\hat{u}(k) = \hat{f}(k).$$

for all $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$. Since u is a strong $\mathscr{Y}$-solution of (P_1^2), then $u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B)) \cap \mathscr{Y}_{per}^{[1]}$, $u' \in \mathscr{Y}(D(\Lambda)) \cap \mathscr{Y}(D(M))$, $Mu' \in \mathscr{Y}_{per}^{[1]}$ and

$$\begin{aligned} &(Mu')'(t) - \Lambda u'(t) - \tfrac{d}{dt}(c * u)(t) \\ &= \gamma_\infty u(t) + Au(t) + (a * Au)(t) + b_\infty Bu(t) + (b * Bu)(t) + f(t), \text{ for a.e } t \in [0, 2\pi]. \end{aligned}$$

Since $u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B))$, then

$$\hat{u}(k) \in D(A) \cap D(B) \text{ and } \widehat{Au}(k) = A\hat{u}(k),\ \hat{Bu}(k) = B\hat{u}(k).$$

by Lemma 1.3.7. Since $u \in \mathscr{Y}_{per}^{[1]}$, then $\widehat{u'}(k) = ik\hat{u}(k)$ by (1.4). Since $u' \in \mathscr{Y}(D(\Lambda)) \cap \mathscr{Y}(D(M))$, then $\widehat{(\Lambda u')} = \Lambda \widehat{u'}(k) = ik\Lambda \hat{u}(k)$, $\widehat{Mu'} = M\widehat{u'}(k) = ikM\hat{u}(k)$ by Lemma 1.3.7. Since $Mu' \in \mathscr{Y}_{per}^{[1]}$, then $\widehat{(Mu')'} = ik\widehat{Mu'}(k) = -k^2 M\hat{u}(k)$ by (1.4). Since $u \in \mathscr{Y}(D(A)) \subset L^1(0, 2\pi; D(A))$, $u \in \mathscr{Y}(D(B)) \subset L^1(0, 2\pi; D(B))$ and a, b, $c \in L^1(\mathbb{R}_+)$, then $c * u$, $a * Au$, $b * Bu \in L^1(0, 2\pi; X)$, $(c * u)(0) = (c * u)(2\pi)$ by (1.2) and $\widehat{(c * u)}(k) = \tilde{c}(ik)\hat{u}(k)$, $\widehat{(a * Au)}(k) = \tilde{a}(ik)A\hat{u}(k)$, $\widehat{(b * Bu)}(k) = \tilde{b}(ik)B\hat{u}(k)$ by (1.3). Given that $\mathscr{Y} \subset L^1(0, 2\pi; X)$, then u, $\Lambda u'$, $(Mu')'$ and $f \in L^1(0, 2\pi; X)$. So u, Au, Bu, $a * Au$, $b * Bu$, $\Lambda u'$, $(Mu')'$ and f all belong to $L^1(0, 2\pi; X)$. Then $\frac{d}{dt}(c * u)$ must belong to $L^1(0, 2\pi; X)$. Therefore $c * u \in W_{per}^{1,1}(0, 2\pi; X)$ and $\widehat{\frac{d}{dt}(c * u)}(k) = ik\tilde{c}(ik)\hat{u}(k)$ by (1.4).

Taking Fourier series on both sides of (1) we obtain that

$$[-k^2 M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_k I - \gamma_\infty I]\hat{u}(k) = \hat{f}(k),\ k \in \mathbb{Z}.$$

$\square$

We adopt the following definition of well-posedness.

Definition 2.1.4. *We say that (TIP_1^2) is $\mathscr{Y}$-well-posed, if for each $f \in \mathscr{Y}$, there exists a unique strong $\mathscr{Y}$-solution u of (P_1^2) which depends continuously on f in the sense that the operator $\mathscr{S} : \mathscr{Y} \to \mathscr{Y}$ defined by $\mathscr{S}(f) = u$ where u is the unique strong $\mathscr{Y}$-solution of (TIP_1^2), is continuous.*

The operator $\mathscr{S}$ is obviously linear.

Remark 2.1.5.

We note that, according to Chapter 1, [6, 7, 15], all the spaces of vector-valued functions $\mathscr{Y}$ concerned in this work are continuously embedded in $L^1(0, 2\pi, X)$. It follows that: If $f_n \to f$ in $\mathscr{Y}$, then $f_n \to f$ in $L^1(0, 2\pi, X)$ and consequently for each $k \in \mathbb{Z}$, $\lim_{n\to\infty} \hat{f}_n(k) = f(k)$ in X. This follows from the dominated convergence theorem.

Our definition imposes an additional condition to that given in the previous works such as [12], [35] that allows us to establish the following characterization of well-posed of (TIP_1^2) in terms of operator-valued Fourier multipliers.

Theorem 2.1.6. *Let X be a Banach space, let A, B, Λ, M be closed linear operators in X such that $D(A)\cap D(B) \subset D(\Lambda)\cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants and a_k, b_k, c_k are defined by (2.2). Then the following assertions are equivalent.*

(i) (TIP_1^2) *is $\mathscr{Y}$-well-posed.*

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ *and* $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, *and* $(kN_k)_{k\in\mathbb{Z}}$ *are $\mathscr{Y}$-Fourier multipliers, where*

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}.$$

In this case the following maximal regularity property holds: The unique strong $\mathscr{Y}$-solution u is such that Au, Bu, $a * Au$, $b * Bu$, Λu, $\Lambda u'$, $c * u$, $\frac{d}{dt}(c * u)$, Mu, Mu' and $(Mu')' \in \mathscr{Y}$ and there exists a constant $C > 0$ independent of $f \in \mathscr{Y}$ such that

$$\begin{aligned} &\|u\|_{\mathscr{Y}} + \|Au\|_{\mathscr{Y}} + \|Bu\|_{\mathscr{Y}} + \|a * Au\|_{\mathscr{Y}} + \|b * Bu\|_{\mathscr{Y}} + \|\Lambda u\|_{\mathscr{Y}} + \|\Lambda u'\|_{\mathscr{Y}} \\ &+\|c * u\|_{\mathscr{Y}} + \|\frac{d}{dt}(c * u)\|_{\mathscr{Y}} + \|Mu\|_{\mathscr{Y}} + \|Mu'\|_{\mathscr{Y}} + \|(Mu')'\|_{\mathscr{Y}} \le C\|f\|_{\mathscr{Y}} \end{aligned}$$

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$. Then $\hat{f}(k) = y$, and $\hat{f}(j) = 0$ if $j \neq k$. By assumption, there exists a unique strong $\mathscr{Y}$-solution u of

(TIP_1^2). By Lemma 2.1.3, we have that for all $k \in \mathbb{Z}$,

$$[-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]\hat{u}(k) = y.$$

It follows that

$$[-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]$$

is surjective for each $k \in \mathbb{Z}$.

Next we prove that for each $k \in \mathbb{Z}$,

$$[-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]$$

is injective for each $k \in \mathbb{Z}$. Let $x \in D(A) \cap D(B)$ such that

$$[-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]x = 0. \tag{2.3}$$

Define $u(t) = e^{ikt}x$ when $t \in [0, 2\pi]$, then $\hat{u}(k) = x$ and $\hat{u}(n) = 0$ for all $n \in \mathbb{Z}$, $n \neq k$. By (2.3) we have that

$$\begin{aligned}\widehat{(Mu')'}(n) - \widehat{\Lambda u'}(n) - \widehat{\frac{d}{dt}(c * u)}(n) &= \gamma_\infty \hat{u}(n) + \widehat{Au}(n) + \widehat{(a * Au)}(n) \\ &\quad + b_\infty \widehat{Bu}(n) + \widehat{(b * Bu)}(n),\end{aligned}$$

for all $n \in \mathbb{Z}$. From the uniqueness theorem of Fourier coefficients, we conclude that u satisfies that

$$\begin{aligned}(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt}(c * u)(t) &= \gamma_\infty u(t) + Aw(t) + (a * Au)(t) \\ &\quad + b_\infty Bu(t) + (b * Bu)(t)\end{aligned}$$

for almost all $t \in [0, 2\pi]$. Thus u is a strong $\mathscr{Y}$-solution of (TIP_1^2) with $f = 0$. We obtain $x = 0$ by the uniqueness assumption. We have shown that

$$[-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]$$

is injective for each $k \in \mathbb{Z}$. Now we show that

$$N_k = [k^2 M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_k I + \gamma_\infty I]^{-1} \in \mathscr{L}(X).$$

Let $k \in \mathbb{Z}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to x$. For each $n \in \mathbb{N}$ we define $f_n(t) = e^{ikt}x_n$ and $f(t) = e^{ikt}x$. Then $f_n, f \in \mathscr{Y}$, for every $n \in \mathbb{N}$ and $f_n \to f$ in $\mathscr{Y}$. Since (TIP_1^2) is $\mathscr{Y}$-well-posed, then for each $f_n, f \in \mathscr{Y}$ there exists a unique strong $\mathscr{Y}$-solution $\mathscr{S}(f_n) = u_n$, $\mathscr{S}(f) = u$. Since $f_n \to f$ in $\mathscr{Y}$, then $u_n \to u$ in $\mathscr{Y}$ by continuity of $\mathscr{S}$. Therefore $\hat{u}_n(k) \to \hat{u}(k)$ by Remark 2.1.5. Since

$$-k^2 M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_k I - \gamma_\infty I$$

is bijective we obtain that $\hat{u}_n(k) = -N_k x_n, \hat{u}(k) = -N_k x$ by Lemma 2.1.3, then $N_k x_n \to N_k x$. Thus, by the Closed Graph Theorem, $N_k \in \mathscr{L}(X)$. Therefore $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$.

We now set for each $k \in \mathbb{Z}$:

$$M_k = k^2 M N_k, \tag{2.4}$$

$$A_k = A N_k, \tag{2.5}$$

$$B_k = B N_k, \tag{2.6}$$

$$H_k = k N_k, \tag{2.7}$$

$$S_k = k\Lambda N_k. \tag{2.8}$$

We will show that $(M_k)_{k\in\mathbb{Z}}$, $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(H_k)_{k\in\mathbb{Z}}$, and $(S_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers. Since $N_k \in \mathscr{L}(X)$ and A, B, Λ, M are closed, then M_k, A_k, B_k, H_k and S_k are bounded for all $k \in \mathbb{Z}$. Now let $f \in \mathscr{Y}$, then there exists a strong $\mathscr{Y}$-solution u of (TIP_1^2). Then $\hat{u}(k) = -N_k \hat{f}(k)$ for all $k \in \mathbb{Z}$ by Lemma 2.1.3. Therefore

$$\hat{u}(k) \in D(A) \cap D(B) \subset D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$. Since A and B are closed, then

$$\widehat{Au}(k) = A\hat{u}(k) = -AN_k\hat{f}(k) = -A_k\hat{f}(k)$$
$$\widehat{Bu}(k) = B\hat{u}(k) = -BN_k\hat{f}(k) = -B_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$ by Lemma 1.3.7. Since Λ, M are closed, $u \in \mathscr{Y}_{per}^{[1]}$, $u' \in \mathscr{Y}(D(\Lambda)) \cap \mathscr{Y}(D(M))$, and $Mu' \in \mathscr{Y}_{per}^{[1]}$, then

$$\widehat{u'}(k) = ik\hat{u}(k) = -ikN_k\hat{f}(k) = -iH_k\hat{f}(k)$$

$$\widehat{\Lambda u'}(k) = \Lambda\widehat{u'}(k) = ik\Lambda\hat{u}(k) = -ik\Lambda N_k\hat{f}(k) = -iS_k\hat{f}(k)$$

$$\widehat{(Mu')'}(k) = ik\widehat{Mu'}(k) = ikM\widehat{u'}(k) = -k^2M\hat{u}(k) = k^2MN_k\hat{f}(k) = M_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$ by (1.4) and Lemma 1.3.7. It follows that $(M_k)_{k\in\mathbb{Z}}$, $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(H_k)_{k\in\mathbb{Z}}$, and $(S_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers. Therefore the implication $(i) \Rightarrow (ii)$ is true.

$(ii) \Rightarrow (i)$. Since $(kN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier, then $(kc_kN_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers as well by Remarks 2.1.1, 1.3.5, and 1.3.12. Since $(k^2MN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, $(kc_kN_k)_{k\in\mathbb{Z}}$, and $(\Lambda N_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers, then $(ikc_kN_k)_{k\in\mathbb{Z}}$, $(c_kN_k)_{k\in\mathbb{Z}}$, $(ikN_k)_{k\in\mathbb{Z}}$, $(ik\Lambda N_k)_{k\in\mathbb{Z}}$, $(\Lambda N_k)_{k\in\mathbb{Z}}$, $(-k^2MN_k)_{k\in\mathbb{Z}}$ $(ikMN_k)_{k\in\mathbb{Z}}$, and $(MN_k)_{k\in\mathbb{Z}}$ are also $\mathscr{Y}$-Fourier multipliers by Remarks 1.3.5 and 1.3.12. From the fact that $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(\Lambda N_k)_{k\in\mathbb{Z}}$, $(MN_k)_{k\in\mathbb{Z}}$, and $(c_kN_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers, then for all $f \in \mathscr{Y}$, we conclude that exist u, v_1, v_2, v_3, v_4, and $v_5 \in \mathscr{Y}$ such that

$$\hat{u}(k) = N_k\hat{f}(k), \tag{2.9}$$

$$\begin{aligned}
\hat{v}_1(k) &= AN_k\hat{f}(k) = A\hat{u}(k) = \widehat{Au}(k),\\
\hat{v}_2(k) &= BN_k\hat{f}(k) = B\hat{u}(k) = \widehat{Bu}(k),\\
\hat{v}_3(k) &= \Lambda N_k\hat{f}(k) = \Lambda\hat{u}(k) = \widehat{\Lambda u}(k),\\
\hat{v}_4(k) &= MN_k\hat{f}(k) = M\hat{u}(k) = \widehat{Mu}(k),\\
\hat{v}_5(k) &= c_kN_k\hat{f}(k) = c_k\hat{u}(k) = \widehat{c*u}(k),
\end{aligned} \tag{2.10}$$

for all $k \in \mathbb{Z}$ by closedness of A, B, Λ, M, and (1.3). Since $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$, then

$$\hat{u}(k) \in D(A) \cap D(B) \subset D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$ by (3.3). Because A, B, Λ, and M are closed, we conclude that

$$u(t) \in D(A) \cap D(B)$$

and $Au(t) = v_1(t)$, $Bu(t) = v_2(t)$, $\Lambda u(t) = v_3(t)$, $Mu(t) = v_4(t)$ and $(c*u)(t) = v_5(t)$ a.e $t \in [0, 2\pi]$ by (3.4) and Lemma 1.3.7 (here we use the fact that $\mathscr{Y} \subset L^p(0, 2\pi; X)$ where p is the corresponding parameter appearing in the definition of $\mathscr{Y}$). Therefore

$$u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B)),$$

and $c*u$, Λu, $Mu \in \mathscr{Y}$. Since $(ikN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier, then there exists $v_6 \in \mathscr{Y}$ such that

$$\hat{v}_6(k) = ikN_k\hat{f}(k) = ik\hat{u}(k) \in D(\Lambda) \cap D(M). \tag{2.11}$$

for all $k \in \mathbb{Z}$. Therefore by (1.4) and (2.11), $u \in \mathscr{Y}_{per}^{[1]}$, $\widehat{u'}(k) = ik\hat{u}(k)$ and

$$\widehat{u'}(k) \in D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$. Since $(ik\Lambda N_k)_{k\in\mathbb{Z}}$ and $(ikMN_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers, then there exist v_7, $v_9 \in \mathscr{Y}$ such that

$$\begin{aligned} \hat{v_7}(k) &= ik\Lambda N_k \hat{f}(k) = \Lambda(ik\hat{u}(k)) = \Lambda\widehat{u'}(k) = \widehat{\Lambda u'}(k), \\ \hat{v}_8(k) &= ikMN_k\hat{f}(k) = M(ik\hat{u}(k)) = M\widehat{u'}(k) = \widehat{Mu'}(k), \end{aligned} \tag{2.12}$$

for al $k \in \mathbb{Z}$. Since Λ and M are closed, then

$$u'(t) \in D(\Lambda) \cap D(M)$$

and $\Lambda u'(t) = v_7(t)$, $Mu'(t) = v_8(t)$ a.e $t \in [0, 2\pi]$ by (3.5) and Lemma 1.3.7 (here once more, we also use the fact that $\mathscr{Y} \subset L^p(0, 2\pi, X)$). Therefore

$$u' \in \mathscr{Y}(D(\Lambda)) \cap \mathscr{Y}(D(M)).$$

Since $(-k^2MN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier, there exists $v_9 \in \mathscr{Y}$ such that

$$\hat{v}_9(k) = -k^2kMN_k\hat{f}(k) = ik(ikM\hat{u}(k)) = ikM\widehat{u'}(k) = ik\widehat{Mu'}(k), \tag{2.13}$$

for all $k \in \mathbb{Z}$ by (3.5). Then $Mu' \in \mathscr{Y}_{per}^{[1]}$. Since $(ikc_kN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier, there exists $v_{10} \in \mathscr{Y}$ such that

$$\hat{v}_{10}(k) = ikc_kN_k\hat{f}(k) = ikc_k\hat{u}(k) = ik\widehat{(c * u)}(k), \tag{2.14}$$

for all $k \in \mathbb{Z}$ by (3.4). Then $c * u \in \mathscr{Y}_{per}^{[1]}$ by (1.4). Now, since $\hat{u}(k) = N_k\hat{f}(k)$, we have

$$[-k^2M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I](-\hat{u}(k)) = \hat{f}(k),$$

this means that

$$\begin{aligned} \widehat{(Mw')}'(k) - \widehat{\Lambda w'}(k) - \widehat{\frac{d}{dt}(c * w)}(k) &= \gamma_\infty\hat{w}(k) + \widehat{Aw}(k) + \widehat{(a * Aw)}(k) \\ &\quad + b_\infty\widehat{Bw}(k) + \widehat{(b * Bw)}(k) + \hat{f}(k), \end{aligned}$$

for all $k \in \mathbb{Z}$ where $w = -u$. From the uniqueness theorem of Fourier coefficients, we conclude that w satisfies

$$(Mw')'(t) - \Lambda w'(t) - \frac{d}{dt}(c * w)(t) = \gamma_\infty w(t) + Aw(t) + (a * Aw)(t) + b_\infty Bw(t) + (b * Bw)(t) + f(t)$$

for almost all $t \in [0, 2\pi]$. Thus w is a strong $\mathscr{Y}$-solution of (TIP_1^2). To prove uniqueness, let u be a strong $\mathscr{Y}$-solution of (TIP_1^2) with $f = 0$. Then

$$[-k^2 M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_k I - \gamma_\infty I]\hat{u}(k) = 0$$

for all $k \in \mathbb{Z}$ by Lemma 2.1.3. Since $ik \in \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ for all $k \in \mathbb{Z}$, then $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. From the uniqueness theorem of Fourier coefficients we have that $u = 0$. Now we show the continuous dependence of u on f. Let $f \in \mathscr{Y}$, then the unique strong $\mathscr{Y}$-solution of (TIP_1^2), u is such that $\hat{u}(k) = -N_k \hat{f}(k)$ for all $k \in \mathbb{Z}$ by Lemma 2.1.3 and $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$. Since N_k is a $\mathscr{Y}$-Fourier multiplier, then there exists a bounded linear operator $T \in \mathscr{L}(\mathscr{Y}, \mathscr{Y})$ such that $\widehat{Tf}(k) = \hat{u}(k)$ for all $k \in \mathbb{Z}$ by Remark 1.3.12. Then $Tf = u$, so u depends continuously of f.

The last assertion of the theorem is a direct consequence of the fact that Au, Bu, $a * Au$, $b * Bu$, Λu, $\Lambda u'$, $c * u$, $\frac{d}{dt}(c * u)$, Mu, Mu' and $(Mu')' \in \mathscr{Y}$ are defined through the operator valued Fourier multipliers $(-AN_k)_{k\in\mathbb{Z}}$, $(-BN_k)_{k\in\mathbb{Z}}$, $(-a_k AN_k)_{k\in\mathbb{Z}}$, $(-b_k BN_k)_{k\in\mathbb{Z}}$, $(-\Lambda N_k)_{k\in\mathbb{Z}}$, $(-k\Lambda N_k)_{k\in\mathbb{Z}}$, $(-c_k N_k)_{k\in\mathbb{Z}}$, $(-kc_k N_k)_{k\in\mathbb{Z}}$, $(-MN_k)_{k\in\mathbb{Z}}$, $(kMN_k)_{k\in\mathbb{Z}}$ and $(k^2 MN_k)_{k\in\mathbb{Z}}$ respectively (here we use the Remarks 2.1.1, 1.3.5 and 1.3.12).

□

The last assertion of the previous theorem is known as the *maximal regularity* property for (TIP_1^2).

Remark 2.1.7.

We can construct the solution $u(\cdot)$ given by the above theorems using Proposition 1.3.6 and the fact that $\mathscr{Y}$ is continuously embedded in $L^p(0,2\pi;X)$. More precisely,

$$u(\cdot) = -\lim_{n\to\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k(\cdot) \otimes N_k \hat{f}(k), \text{ with convergence in } L^p(0,2\pi;X). \tag{2.15}$$

Remark 2.1.8.

If at most one operator of those that appear in (1) is unbounded, then the additional condition in our definition of well posedness is obtained automatically. In that case the operators $-k^2M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I$ are closed for all $k \in \mathbb{Z}$ and once we show that they are bijective, continuity follows from the Closed Graph Theorem.

2.2 Characterization of Maximal Regularity on Periodic Lebesgue, Besov and Triebel-Lizorkin Spaces for the Equation of Type I

In this section, we give concrete conditions that allow us to apply Theorem 2.1.6. Specifically we obtain conditions under which the sequences $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are Fourier multipliers in the scale of spaces under consideration by use of the operator valued multiplier theorems established in [4], [6], [7], [15]. Versions of the multiplier theorems on the real line can be found in [2], [27], [28] (this reference contains criteria for $R-$boundedness of operator families), [45], [46]. The L^p-case is much different from the other scales of spaces in that it involves the notion of R-boundedness and one has to restrict consideration to UMD Banach spaces. Fortunately, many Banach spaces, for example Hilbert spaces and $L^p(\Omega,\mu)$, $1 < p < \infty$ are UMD spaces. In addition, the R-boundedness condition holds for resolvents of many classical operators in the analysis of partial differential equations of evolution type (see for example Kunstmann-Weis [33] and Girardi-Weis [28]).

Let $\{d_k : k \in \mathbb{Z}\}$ be a scalar sequence. We will use the following hypotheses:

(H0): $\{d_k : k \in \mathbb{Z}\}$ is bounded.

(H1): $\{d_k : k \in \mathbb{Z}\}$, $\{k\Delta d_k : k \in \mathbb{Z}\}$ are bounded.

(H2): $\{d_k : k \in \mathbb{Z}\}$, $\{k\Delta d_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 d_k : k \in \mathbb{Z}\}$ are bounded.

(H3): $\{d_k : k \in \mathbb{Z}\}$, $\{k\Delta d_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 d_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 d_k : k \in \mathbb{Z}\}$ are bounded.

Clearly (H0) is weaker than (H1) which in turn is weaker than (H2), and the later weaker that (H3). In our cases (H0) is obtained automatically from the Riemann-Lebesgue Lemma. The condition (H1) will be used for L^p well-posedness, while (H2) and (H3) are needed for Besov spaces and Triebel-Lizorkin spaces respectively. Some variations to this rule will occur when the Banach space X satisfies a special geometric property such as being UMD or having nontrivial Fourier type.

Examples of functions $d(t)$ such that $d_k = \tilde{d}(ik)$ satisfies (H3) are $d(t) = Ce^{-\omega t}t^\nu$ where $\omega > 0$, $\nu > -1$ and C is a constant. Now let $\beta > 0$ and consider the family of functions $d(t) = \begin{cases} 0 & \text{if } 0 < t < \beta, \\ Ce^{-\omega t}(t-\beta)^\nu & \text{if } t > \beta \end{cases}$ where $\omega > 0$, $\nu > -1$ and C is a constant and $d_k = \tilde{d}(ik)$. The sequence d_k satisfies (H0) automatically from Riemann-Lebesgue Lemma. If $-1 < \nu < 0$ and β is not a multiple of 2π, then d_k does not satisfy (H1). If $0 \le \nu < 1$ and β is not a multiple of 2π, then d_k satisfies (H1) but not (H2). If $1 \le \nu < 2$ and β is not a multiple of 2π, then d_k satisfies (H2) but not (H3). If $\nu \ge 2$ or β is a multiple of 2π, then d_k satisfies (H3).

Theorem 2.2.1. *Let X be a UMD Banach space, $1 < p < \infty$ and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$, and $\{c_k : k \in \mathbb{Z}\}$ satisfies (H1), where a_k, b_k, c_k are defined by (2.2). Then the following assertions are equivalent.*

(i) (TIP_1^2) is L^p-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, *and* $\{kN_k : k \in \mathbb{Z}\}$ *are R-bounded, where*

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \implies (ii)$ Assume that (TIP_1^2) is L^p-well-posed. Then by Theorem 2.1.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. The R-boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $\{kN_k : k \in \mathbb{Z}\}$ now follows from Proposition 1.4.3.

$(ii) \implies (i)$ In view of Theorem 2.1.6, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

For each $k \in \mathbb{Z}$ we define $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, $H_k = kN_k$ and $S_k = k\Lambda N_k$. These operators are bounded because $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$. Since $\{kN_k : k \in \mathbb{Z}\}$ is R-bounded, $\{N_k : k \in \mathbb{Z}\}$ is R-bounded by Remark 1.4.5.

Now we note the following equality,

$$M_k + (1 + a_k)A_k + (b_\infty + b_k)B_k + iS_k + ic_kH_k + \gamma_\infty N_k = -I.$$

which implies that

$$A_k = -\frac{1}{1 + a_k}[I + (M_k + (b_\infty + b_k)B_k + iS_k + ic_kH_k + \gamma_\infty N_k].$$

for each $k \in \mathbb{Z}$ such that $a_k \neq -1$. Since $\{M_k : k \in \mathbb{Z}\}$, $\{B_k : k \in \mathbb{Z}\}$, $\{H_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are R-bounded, then the sequence $\{A_k : k \in \mathbb{Z}\}$ is also R-bounded by Remark 1.4.5 and Remark 2.1.1.

We observe that

$$
\begin{aligned}
&N_{k+1}^{-1}N_k \\
&= [(k+1)^2M + (1+a_k)A + (b_\infty + b_k)B + i(k+1)\Lambda + i(k+1)c_{k+1}I + \gamma_\infty I]N_k \\
&= [N_k^{-1} + (2k+1)M + \Delta a_k A + \Delta b_k B + ik\Delta c_k I + ic_{k+1}I + i\Lambda]N_k \\
&= I + (2k+1)MN_k + \Delta a_k AN_k + \Delta b_k BN - k + ik\Delta c_k N_k + ic_{k+1}N_k + i\Lambda N_k \\
&= I + \frac{2k+1}{k^2}M_k + \Delta a_k A_k + \Delta b_k B_k + i\Delta c_k H_k + \frac{ic_{k+1}}{k}H_k + \frac{i}{k}S_k
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0$.

If we define

$$
T_k = \frac{2k+1}{k^2}M_k + \Delta a_k A_k + \Delta b_k B_k + i\Delta c_k H_k + i\frac{c_{k+1}}{k}H_k + \frac{i}{k}S_k, \tag{2.16}
$$

then $N_{k+1}^{-1}N_k = I + T_k$ for all $k \in \mathbb{Z}$, $k \neq 0$.

Define

$$
Q_k = -kT_k = -[\frac{2k+1}{k}M_k + k\Delta a_k A_k + k\Delta b_k B_k + ik\Delta c_k H_k + ic_{k+1}H_k + iS_k].
$$

for all $k \in \mathbb{Z}$, $k \neq 0$. Since $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$, and $\{c_k : k \in \mathbb{Z}\}$ satisfy (H1), then $\{Q_k : k \in \mathbb{Z}\}$ is R-bounded by Remark 1.4.5.

We observe that

$$
\begin{aligned}
k\Delta N_k &= k(N_{k+1} - N_k) \\
&= kN_{k+1}(I - N_{k+1}^{-1}N_k) \\
&= kN_{k+1}[I - (I + T_k)] \\
&= kN_{k+1}[-T_k] \\
&= N_{k+1}Q_k.
\end{aligned}
$$

Thus, we have

$$k\Delta A_k = k\Delta(AN_k) = Ak\Delta N_k = AN_{k+1}Q_k = A_{k+1}Q_k,$$

$$k\Delta B_k = k\Delta(BN_k) = B(k\Delta N_k) = BN_{k+1}Q_k = B_{k+1}Q_k,$$

$$\begin{aligned}
k\Delta H_k &= k[(k+1)N_{k+1} - kN_k] \\
&= k[(k+1)N_{k+1} - (k+1)N_k + (k+1)N_k - k\Lambda N_k] \\
&= k[(k+1)\Delta N_k + N_k] \\
&= (k+1)(k\Delta N_k) + kN_k \\
&= (k+1)N_{k+1}Q_k + kN_k \\
&= H_{k+1}Q_k + H_k,
\end{aligned}$$

$$\begin{aligned}
k\Delta S_k &= \Lambda(k[(k+1)N_{k+1} - kN_k]) \\
&= \Lambda[H_{k+1}Q_k + H_k] \\
&= S_{k+1}Q_k + S_k,
\end{aligned}$$

and

$$\begin{aligned}
k\Delta M_k &= k((k+1)^2 MN_{k+1} - k^2 MN_k) \\
&= k((k+1)^2 MN_{k+1} - (k+1)^2 MN_k + (k+1)^2 MN_k - k^2 MN_k) \\
&= k[(k+1)^2 M\Delta N_k + (2k+1)MN_k \\
&= (k+1)^2 M[k\Delta N_k] + k(2k+1)MN_k \\
&= (k+1)^2 MN_{k+1}Q_k + k(2k+1)MN_k \\
&= M_{k+1}Q_k + \frac{2k+1}{k}M_k
\end{aligned}$$

for all $k \in \mathbb{Z}$, $k \neq 0$.

Then $\{k\Delta A_k : k \in \mathbb{Z}\}$, $\{k\Delta B_k : k \in \mathbb{Z}\}$, $\{k\Delta H_k : k \in \mathbb{Z}\}$, $\{k\Delta S_k : k \in \mathbb{Z}\}$ and $\{k\Delta M_k : k \in \mathbb{Z}\}$ are R-bounded by Remark 1.4.5. Therefore by Theorem 1.4.2 we obtain that $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(H_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ and $(M_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

□

From the proof of Theorem 2.2.1, we deduce the following result for $B^s_{pq}-$solutions in case X has nontrivial Fourier type.

Theorem 2.2.2. *Let X be a Banach space with nontrivial Fourier type and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$, and $(c_k)_{k\in\mathbb{Z}}$ satisfies (H1), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 \le p, q \le \infty$, the following are equivalent.*

(i) (TIP^2_1) is B^s_{pq}-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ are bounded, where

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \implies (ii)$. Assume that (TIP^2_1) is B^s_{pq}-well-posed. Then by Theorem 2.1.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. The boundedness of $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ now follows from Remark 1.3.12.

$(ii) \implies (i)$. In view of Theorem 2.1.6, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. By Theorem 1.3.10 the proof follows the same lines as that of the preceding theorem.

□

We now consider the problem of well-posedness in Besov spaces $B^s_{pq}(0, 2\pi, X)$ for arbitrary Banach spaces X. For this, assumption (H1) are no longer sufficient. It is proved in [6, Theorem 4.2] that for any sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathscr{L}(X)$, the so-called variational Marcinkiewicz condition, that is

$$\sup_{k\in\mathbb{Z}} \|M_k\| + \sup_{j\geq 0} \left(\sum_{2^j\leq |k|<2^{j+1}} \|\Delta M_k\| \right) < \infty \tag{2.17}$$

implies that $(M_k)_{k\in\mathbb{Z}}$ is a B^s_{pq}-Fourier multiplier if and only if $1 < p < \infty$ and X is a UMD space.

For Banach spaces with nontrivial Fourier type, a condition which implies that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier for the scale $B^s_{p,q}$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$ is the Marcinkiewicz condition of order one:

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\|) < \infty, \tag{2.18}$$

see Theorem 1.3.10, which is used in the proof of Theorem 2.2.2.

For arbitrary Banach spaces, a Marcinkiewicz condition of order two is needed, namely,

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\|) < \infty, \tag{2.19}$$

see Theorem 1.3.10.

Our next result uses this condition to obtain maximal regularity of (TIP^2_1) when X does not necessarily have nontrivial Fourier type.

Theorem 2.2.3. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$, and $(c_k)_{k\in\mathbb{Z}}$ satisfy (H2), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 \leq p, q \leq \infty$, the following statements are equivalent.*

(i) (TIP^2_1) is B^s_{pq}-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, *and* $\{kN_k : k \in \mathbb{Z}\}$ *are bounded, where*

$$N_k = [k^2M + (1+a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \implies (ii)$. Assume that (TIP_1^2) is B^s_{pq}-well-posed. Then by Theorem 2.1.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$ and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{kN_k : k \in \mathbb{Z}\}$, and $\{kN_k : k \in \mathbb{Z}\}$ now follows of Remark 1.3.12.

$(ii) \implies (i)$. In view of Theorem 2.1.6, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers.

Let $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, $H_k = kN_k$, and $S_k = k\Lambda N_k$. Since (H2) implies (H1), then the verification of the Marcinkiewicz condition of order one is similar to what was done in the proof of Theorem 2.2.1. It remains to prove that $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 M_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 A_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 B_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 S_k\| < \infty$, and $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 N_k\| < \infty$.

We recall from the proof of Theorem 2.2.1 that the family $(T_k)_{k\in\mathbb{Z}}$ defined through

$$T_k = \frac{2k+1}{k^2}M_k + \Delta a_k A_k + \Delta b_k B_k + i\Delta c_k H_k + i\frac{c_{k+1}}{k}H_k + i\frac{1}{k}S_k,\ k \neq 0$$

is such that $N_{k+1}^{-1}N_k = I + T_k$, $Q_k = -kT_k$, $k\Delta N_k = N_{k+1}Q_k$ for all $k \in \mathbb{Z}$, $k \neq 0$, and $\{kT_k : k \in \mathbb{Z}\}$ is bounded.

We observe that

$$\Delta T_k = \Delta(\frac{2k+1}{k^2}M_k) + \Delta[(\Delta a_k)A_k] + \Delta(\Delta b_k)B_k + i\Delta(\Delta(c_k)H_k) + i\Delta(\frac{c_{k+1}}{k}H_k) + i\Delta(\frac{1}{k}S_k)).$$

We will consider each term separately:

$$\begin{aligned}
\Delta(\frac{2k+1}{k^2}M_k) &= \frac{2k+3}{(k+1)^2}M_{k+1} - \frac{2k+1}{k^2}M_k \\
&= \frac{2k+3}{(k+1)^2}M_{k+1} - \frac{2k+3}{(k+1)^2}M_k + \frac{2k+3}{(k+1)^2}M_k - \frac{2k+1}{k^2}M_k \\
&= \frac{2k+3}{(k+1)^2}\Delta M_k - \frac{2k^2+4k+1}{k^2(k+1)^2}M_k \\
&= \frac{2k+3}{k(k+1)^2}(k\Delta M_k) - \frac{2k^2+4k+1}{k^2(k+1)^2}M_k,
\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{1}{k}S_k) &= \frac{1}{k+1}S_{k+1} - \frac{1}{k}S_k \\
&= \frac{1}{k+1}S_{k+1} - \frac{1}{k+1}S_k + \frac{1}{k+1}S_k - \frac{1}{k}S_k \\
&= \frac{1}{k+1}\Delta S_k - \frac{1}{k(k+1)}S_k \\
&= \frac{1}{k(k+1)}(k\Delta S_k) - \frac{1}{k(k+1)}S_k,
\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{c_{k+1}}{k}H_k) &= \frac{c_{k+2}}{k+1}H_{k+1} - \frac{c_{k+1}}{k}H_k \\
&= \frac{c_{k+2}}{k+1}H_{k+1} - \frac{c_{k+2}}{k+1}H_k + \frac{c_{k+2}}{k+1}H_k - \frac{c_{k+2}}{k}H_k + \frac{c_{k+2}}{k}H_k - \frac{c_{k+1}}{k}H_k \\
&= \frac{c_{k+2}}{k+1}\Delta H_k + \frac{\Delta c_{k+1}}{k}H_k - \frac{c_{k+2}}{k(k+1)}H_k \\
&= \frac{c_{k+2}}{k(k+1)}(k\Delta H_k) + \frac{(k+1)\Delta c_{k+1}}{k(k+1)}H_k - \frac{c_{k+2}}{k(k+1)}H_k,
\end{aligned}$$

$$\begin{aligned}
\Delta[(\Delta a_k)A_k] &= (\Delta a_{k+1})A_{k+1} - (\Delta a_k)A_k \\
&= (\Delta a_{k+1})A_{k+1} - (\Delta a_{k+1})A_k + (\Delta a_{k+1})A_k - (\Delta a_k)A_k \\
&= (\Delta a_{k+1})\Delta A_k + (\Delta^2 a_k)A_k \\
&= \frac{1}{k(k+1)}((k+1)\Delta a_{k+1})(k\Delta A_k) + \frac{1}{k^2}(k^2\Delta^2 a_k)A_k,
\end{aligned}$$

$$\begin{aligned}\Delta[k(\Delta b_k)B_k] &= (\Delta b_{k+1})B_{k+1} - (\Delta b_k)B_k \\ &= (\Delta b_{k+1})B_{k+1} - (\Delta b_{k+1})B_k + (\Delta b_{k+1})B_k - (\Delta b_k)B_k \\ &= (\Delta b_{k+1})\Delta B_k + (\Delta^2 b_k)B_k \\ &= \frac{1}{k(k+1)}((k+1)\Delta b_{k+1})(k\Delta B_k) + \frac{1}{k^2}(k^2\Delta^2 b_k)B_k,\end{aligned}$$

and

$$\begin{aligned}\Delta((\Delta c_k)H_k)) &= (\Delta c_{k+1})H_{k+1} - (\Delta c_k)H_k \\ &= (\Delta c_{k+1})H_{k+1} - (\Delta c_{k+1})H_k + (\Delta c_{k+1})H_k - (\Delta c_k)H_k \\ &= (\Delta c_{k+1})\Delta H_k + (\Delta^2 c_k)H_k \\ &= \frac{1}{k(k+1)}((k+1)\Delta c_{k+1})(k\Delta H_k) + \frac{1}{k^2}(k^2\Delta^2 c_k)H_k\end{aligned}$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1$.

Since $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$, $\{c_k : k \in \mathbb{Z}\}$ satisfy (H2), $(M_k)_{k\in\mathbb{Z}}$, $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$ satisfy the Marcinkiewicz condition of order one, and $\{c_k : k \in \mathbb{Z}\}$ is bounded by Remark 2.1.1. It follows that $\sup_{k\in\mathbb{Z}}\{k^2\|\Delta T_k\|\} < \infty$.

We observe that from (2.16) that

$$\begin{aligned}k^2\Delta^2 N_k &= k^2[\Delta N_{k+1} - \Delta N_k)] \\ &= k^2[N_{k+2}(I - (I + T_{k+1})) - N_{k+1}(I - (I + T_k))] \\ &= k^2[-N_{k+2}T_{k+1} + N_{k+1}T_k] \\ &= -k^2 N_{k+2}[T_{k+1} - N_{k+2}^{-1}N_{k+1}T_k] \\ &= -k^A N_{k+2}[T_{k+1} - (I + T_{k+1})T_k] \\ &= -k^2 N_{k+2}[T_{k+1} - T_k - T_{k+1}T_k] \\ &= -k^2 N_{k+2}[\Delta T_k - T_{k+1}T_k] \\ &= -N_{k+2}[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k] = N_{k+2}R_k\end{aligned}$$

where we have set $R_k = -[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k]$ for all $k \in \mathbb{Z}$, $k \neq 0, -1$. Since $\{Q_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta T_k : k \in \mathbb{Z}\}$ are bounded, then $\{R_k : k \in \mathbb{Z}\}$ is bounded.

Now, we have that

$$k^2\Delta^2 A_k = k^2\Delta^2(AN_k) = A(k^2\Delta^2 N_k) = AN_{k+2}R_k = A_{k+2}R_k,$$

$$k^2\Delta^2 B_k = k^2\Delta^2(BN_k) = B(k^2\Delta^2 N_k) = BN_{k+2}R_k = B_{k+2}R_k,$$

$$\begin{aligned}
k^2\Delta^2 H_k &= k^2\Delta^2(kN_k) \\
&= k^2[(k+2)N_{k+2} - 2(k+1)N_{k+1} + kN_k] \\
&= k^2[kN_{k+2} - 2kN_{k+1} + kN_k] + 2k^2N_{k+2} - 2k^2N_{k+1} \\
&= k^3\Delta^2 N_k + 2k^2\Delta N_{k+1} \\
&= k(k^2\Delta^2 N_k) + \frac{2k^2}{k+1}[(k+1)\Delta N_{k+1}] \\
&= kN_{k+2}R_k + \frac{2k^2}{k+1}N_{k+2}Q_{k+1} \\
&= \frac{k}{k+2}H_{k+2}R_k + \frac{2k^2}{(k+1)(k+2)}H_{k+2}Q_{k+1},
\end{aligned}$$

$$\begin{aligned}
k^2\Delta^2 S_k &= k^2\Delta^2(k\Lambda N_k) \\
&= k^2\Lambda\Delta^2(kN_k) \\
&= \Lambda(k^2\Delta^2 H_k) \\
&= \Lambda\left(\frac{k}{k+2}H_{k+2}R_k + \frac{2k^2}{(k+1)(k+2)}H_{k+2}Q_{k+1}\right) \\
&= \frac{k}{k+2}S_{k+2}R_k + \frac{2k^2}{(k+1)(k+2)}S_{k+2}Q_{k+1}.
\end{aligned}$$

where we assume $k \notin \{0, -1, -2\}$.

Finally,

$$
\begin{aligned}
k^2\Delta^2 M_k &= k^2\Delta^2(k^2 MN_k)\\
&= k^2[(k+2)^2 MN_{k+2} - 2(k+1)^2 MN_{k+1} + k^2 MN_k]\\
&= k^2[k^2 MN_{k+2} - 2k^2 MN_{k+1} + k^2 MN_k] + k^2(4k+4)MN_{k+2}\\
&- 2k^2(2k+1)MN_{k+1}\\
&= k^2 M(k^2\Delta^2 N_k) + \frac{2k^2(2k+1)}{k+1}M[(k+1)\Delta N_{k+1}] + 2k^2 MN_{k+2}\\
&= k^2 MN_{k+2}R_k + \frac{2k^2(2k+1)}{k+1}MN_{k+2}Q_{k+1} + 2k^2 MN_{k+2}\\
&= \frac{k^2}{(k+2)^2}M_{k+2}R_k + \frac{2k^2(2k+1)}{(k+1)(k+2)^2}M_{k+2}Q_{k+1} + \frac{2k^2}{(k+2)^2}M_{k+2}
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2$.

Since $\{A_k : k \in \mathbb{Z}\}$, $\{B_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, $\{H_k : k \in \mathbb{Z}\}$, $\{M_k : k \in \mathbb{Z}\}$, $\{Q_k : k \in \mathbb{Z}\}$, and $\{R_k : k \in \mathbb{Z}\}$ are bounded, then $\{k^2\Delta^2 A_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 B_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 H_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 S_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta^2 M_k : k \in \mathbb{Z}\}$ are bounded. This completes the proof.

□

From the proof of Theorem 2.2.3 and using Theorem 1.3.11, we deduce the following result for $F^s_{pq}-$solutions in the case that $1 < p < \infty$, $1 < q \leq \infty$ and $s > 0$.

Theorem 2.2.4. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$, and $(c_k)_{k\in\mathbb{Z}}$ satisfies (H2), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 < p < \infty$, $1 < q \leq \infty$, the following are equivalent.*

(i) (TIP^2_1) is $F^s_{p,q}$-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, *and* $\{kN_k : k \in \mathbb{Z}\}$ *are bounded, where*

$$N_k = [k^2M + (1+a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \implies (ii)$. Follows from Theorem 2.1.6 and Remark 1.3.12.

$(ii) \implies (i)$. Follows from Theorem 1.3.11 using the same lines as the proof of the preceding theorem.

□

We now consider the problem of well-posedness in Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi, X)$ with parameters $1 \le p < \infty$, $1 \le q \le \infty$ and $s > 0$. For this, assumption (H2) is no longer sufficient.

A condition which implies that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier for the scale F^s_{pq}, $s \in \mathbb{R}$, $1 < p < \infty$, $1 < q \le \infty$ is the Marcinkiewicz condition of order two which is used in the proof of Theorem 2.2.4.

For $1 \le p < \infty$, $1 \le q \le \infty$ and $s \in \mathbb{R}$, a Marcinkiewicz condition of order three is needed, namely,

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\| + |k|^3\|\Delta^3 M_k\|) < \infty. \tag{2.20}$$

Our next result uses this condition to obtain characterization of F^s_{pq}-well-posedness of the Problem (TIP^2_1).

Theorem 2.2.5. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $(a_k)_{k\in\mathbb{Z}}$, $(b_k)_{k\in\mathbb{Z}}$, and $(c_k)_{k\in\mathbb{Z}}$ satisfy (H3), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 \le p < \infty$, $1 \le q \le \infty$, the following assertions are equivalent.*

(i) (TIP^2_1) *is* $F^s_{p,q}$*-well-posed.*

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, *and* $\{kN_k : k \in \mathbb{Z}\}$ *are bounded, where*

$$N_k = [k^2M + (1+a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \Longrightarrow (ii)$. Assume that (TIP_1^2) is F^s_{pq}-well-posed. Then by Theorem 2.1.6, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are F^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $\{kN_k : k \in \mathbb{Z}\}$ follows of Remark 1.3.12.

$(ii) \Longrightarrow (i)$. In view of Theorem 2.1.6, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(kN_k)_{k\in\mathbb{Z}}$ are F^s_{pq}-Fourier multipliers. Let $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, $H_k = kN_k$ and $S_k = k\Lambda N_k$. Since (H3) implies (H2), then the verification of the Marcinkiewicz condition of order two is identical to what was done in the proof of Theorem 2.2.3.

It remains to prove that $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 M_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 A_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 B_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 H_k\| < \infty$, and $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 S_k\| < \infty$.

We recall from the proof of Theorem 2.2.3 that

$$T_k = \frac{2k+1}{k^2}M_k + \Delta a_k A_k + \Delta b_k B_k + i\Delta c_k H_k + i\frac{1}{k}c_{k+1}H_k + i\frac{1}{k}S_k,$$

is such that $N_{k+1}^{-1}N_k = I + T_k$, $k\Delta N_k = N_{k+1}Q_k$ where $Q_k = -kT_k$, $k^2\Delta^2 N_k = N_{k+2}R_k$ where $R_k = -[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k]$ for all $k \in \mathbb{Z}$, $k \neq 0, -1$ and the sets $\{Q_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 T_k : k \in \mathbb{Z}\}$, $\{R_k : k \in \mathbb{Z}\}$ are bounded. Also we showed in the proof of Theorem 2.2.3 that

$$\begin{aligned}\Delta T_k = {} & \frac{2k+3}{(k+1)^2}\Delta M_k - \frac{2k^2+4k+1}{k^2(k+1)^2}M_k + i\frac{c_{k+2}}{k+1}\Delta H_k + i\frac{\Delta c_{k+1}}{k}H_k - i\frac{c_{k+2}}{k(k+1)}H_k \\ & + i\frac{1}{k+1}\Delta S_k - i\frac{1}{k(k+1)}S_k + (\Delta a_{k+1})\Delta A_k + (\Delta^2 a_k)A_k + (\Delta b_{k+1})\Delta B_k \\ & + (\Delta^2 b_k)B_k + i(\Delta c_{k+1})\Delta H_k + i(\Delta^2 c_k)H_k.\end{aligned}$$

$k \notin \{0, -1\}$.

Then

$$\begin{aligned}\Delta^2 T_k &= \Delta(\frac{2k+3}{(k+1)^2}\Delta M_k) - \Delta(\frac{2k^2+4k+1}{k^2(k+1)^2}M_k) + i\Delta(\frac{c_{k+2}}{k+1}\Delta H_k)\\ &+ i\Delta(\frac{\Delta c_{k+1}}{k}H_k) - i\Delta(\frac{c_{k+2}}{k(k+1)}H_k) + i\Delta(\frac{1}{k+1}\Delta S_k)\\ &- i\Delta(\frac{1}{k(k+1)}S_k) + \Delta((\Delta a_{k+1})\Delta A_k) + \Delta((\Delta^2 a_k)A_k) + \Delta((\Delta b_{k+1})\Delta B_k)\\ &+ \Delta((\Delta^2 b_k)B_k) + i\Delta((\Delta c_{k+1})\Delta H_k) + i\Delta((\Delta^2 c_k)H_k).\end{aligned}$$

We will consider each term separately:

$$\begin{aligned}\Delta(\frac{2k+3}{(k+1)^2}\Delta M_k) &= \frac{2k+5}{(k+2)^2}\Delta M_{k+1} - \frac{2k+3}{(k+1)^2}\Delta M_k\\ &= \frac{2k+5}{(k+2)^2}\Delta M_{k+1} - \frac{2k+5}{(k+2)^2}\Delta M_k + \frac{2k+5}{(k+2)^2}\Delta M_k - \frac{2k+3}{(k+1)^2}\Delta M_k\\ &= \frac{2k+5}{(k+2)^2}\Delta^2 M_k - \frac{2k^2+4k+1}{(k+1)^2(k+2)^2}\Delta M_k\\ &= \frac{2k+5}{k^2(k+2)^2}(k^2\Delta^2 M_k) - \frac{2k^2+4k+1}{k(k+1)^2(k+2)^2}(k\Delta M_k),\end{aligned}$$

$$\begin{aligned}\Delta(\frac{2k^2+4k+1}{k^2(k+1)^2}M_k) &= \frac{2(k+1)^2+4(k+1)+1}{(k+1)^2(k+2)^2}M_{k+1} - \frac{2k^2+4k+1}{k^2(k+1)^2}M_k\\ &= \frac{2k^2+8k+7}{(k+1)^2(k+2)^2}M_{k+1} - \frac{2k^2+8k+7}{(k+1)^2(k+2)^2}M_k\\ &+ \frac{2k^2+8k+7}{(k+1)^2(k+2)^2}M_k - \frac{2k^2+4k+1}{k^2(k+1)^2}M_k\\ &= \frac{2k^2+8k+7}{(k+1)^2(k+2)^2}\Delta M_k - \frac{2(2k^3+9k^2+10k+2)}{k^2(k+1)^2(k+2)^2}M_k\\ &= \frac{2k^2+8k+7}{k(k+1)^2(k+2)^2}(k\Delta M_k) - \frac{2(2k^3+9k^2+10k+2)}{k^2(k+1)^2(k+2)^2}M_k,\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{c_{k+2}}{k+1}\Delta H_k) &= \frac{c_{k+3}}{k+2}\Delta H_{k+1} - \frac{c_{k+2}}{k+1}\Delta H_k \\
&= \frac{c_{k+3}}{k+2}\Delta H_{k+1} - \frac{c_{k+3}}{k+2}\Delta H_k \\
&+ \frac{c_{k+3}}{k+2}\Delta H_k - \frac{c_{k+2}}{k+2}\Delta H_k + \frac{c_{k+2}}{k+2}\Delta H_k - \frac{c_{k+2}}{k+1}\Delta H_k \\
&= \frac{c_{k+3}}{k+2}\Delta^2 H_k + \frac{\Delta c_{k+2}}{k+2}\Delta H_k - \frac{2c_{k+1}}{(k+1)(k+2)}\Delta H_k \\
&= \frac{c_{k+3}}{k^2(k+2)}(k^2\Delta^2 H_k) + \frac{(k+2)\Delta c_{k+2}}{k(k+2)^2}(k\Delta H_k) - \frac{2c_{k+1}}{k(k+1)(k+2)}(k\Delta H_k),
\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{\Delta c_{k+1}}{k}H_k) &= \frac{\Delta c_{k+2}}{k+2}H_{k+1} - \frac{\Delta c_{k+1}}{k+1}H_k \\
&= \frac{\Delta c_{k+2}}{k+2}H_{k+1} - \frac{\Delta c_{k+2}}{k+2}H_k + \frac{\Delta c_{k+2}}{k+2}H_k - \frac{\Delta c_{k+1}}{k+2}H_k + \frac{\Delta c_{k+1}}{k+2}H_k - \frac{\Delta c_{k+1}}{k+1}H_k \\
&= \frac{\Delta c_{k+2}}{k+2}\Delta H_k + \frac{\Delta^2 c_{k+1}}{k+2}H_k - \frac{\Delta c_{k+1}}{(k+1)(k+2)}H_k \\
&= \frac{(k+2)\Delta c_{k+2}}{k(k+2)^2}(k\Delta H_k) + \frac{(k+1)^2\Delta^2 c_{k+1}}{(k+1)^2(k+2)}H_k - \frac{(k+1)\Delta c_{k+1}}{(k+1)^2(k+2)}H_k,
\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{c_{k+2}}{k(k+1)}H_k) &= \frac{c_{k+3}}{(k+1)(k+2)}H_{k+1} - \frac{c_{k+2}}{k(k+1)}H_k \\
&= \frac{c_{k+3}}{(k+1)(k+2)}H_{k+1} - \frac{c_{k+3}}{(k+1)(k+2)}H_k + \frac{c_{k+3}}{(k+1)(k+2)}H_k \\
&- \frac{c_{k+2}}{(k+1)(k+2)}H_k + \frac{c_{k+2}}{(k+1)(k+2)}H_k - \frac{c_{k+2}}{k(k+1)}H_k \\
&= \frac{c_{k+3}}{(k+1)(k+2)}\Delta H_k + \frac{\Delta c_{k+2}}{(k+1)(k+2)}H_k - \frac{2c_{k+2}}{k(k+1)(k+2)}H_k \\
&= \frac{c_{k+3}}{k(k+1)(k+2)}(k\Delta H_k) + \frac{(k+2)\Delta c_{k+2}}{(k+1)(k+2)^2}H_k - \frac{2c_{k+2}}{k(k+1)(k+2)}H_k,
\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{1}{k+1}\Delta S_k) &= \frac{1}{k+2}\Delta S_{k+1} - \frac{1}{k+1}\Delta S_k \\
&= \frac{1}{k+2}\Delta S_{k+1} - \frac{1}{k+2}\Delta S_k + \frac{1}{k+2}\Delta S_k - \frac{1}{k+1}\Delta S_k \\
&= \frac{1}{k+2}\Delta^2 S_k - \frac{1}{(k+1)(k+2)}\Delta S_k \\
&= \frac{1}{k^2(k+2)}(k^2\Delta^2 S_k) - \frac{1}{k(k+1)(k+2)}(k\Delta S_k),
\end{aligned}$$

$$\begin{aligned}
\Delta(\frac{1}{k(k+1)}S_k) &= \frac{1}{(k+1)(k+2)}S_{k+1} - \frac{1}{k(k+1)}S_k \\
&= \frac{1}{(k+1)(k+2)}S_{k+1} - \frac{1}{(k+1)(k+2)}S_k + \frac{1}{(k+1)(k+2)}S_k - \frac{1}{k(k+1)}S_k \\
&= \frac{1}{(k+1)(k+2)}\Delta S_k - \frac{2}{k(k+1)(k+2)}S_k \\
&= \frac{1}{k(k+1)(k+2)}(k\Delta S_k) - \frac{2}{k(k+1)(k+2)}S_k,
\end{aligned}$$

$$\begin{aligned}
\Delta((\Delta a_{k+1})\Delta A_k) &= (\Delta a_{k+2})\Delta A_{k+1} - \Delta a_{k+1}\Delta A_k \\
&= (\Delta a_{k+2})\Delta A_{k+1} - (\Delta a_{k+2})\Delta A_k + (\Delta a_{k+2})\Delta A_k - \Delta a_{k+1}\Delta A_k \\
&= (\Delta a_{k+2})\Delta^2 A_k + (\Delta^2 a_{k+1})\Delta A_k \\
&= \frac{1}{k^2(k+2)}((k+2)\Delta a_{k+2})(k^2\Delta^2 A_k) + \frac{1}{k(k+1)^2}((k+1)^2\Delta^2 a_{k+1})(k\Delta A_k),
\end{aligned}$$

$$\begin{aligned}
\Delta((\Delta^2 a_k)A_k) &= (\Delta^2 a_{k+1})A_{k+1} - (\Delta^2 a_k)A_k \\
&= (\Delta^2 a_{k+1})A_{k+1} - (\Delta^2 a_{k+1})A_k + (\Delta^2 a_{k+1})A_k - (\Delta^2 a_k)A_k \\
&= (\Delta^2 a_{k+1})\Delta A_k + (\Delta^3 a_{k+1})A_k \\
&= \frac{1}{k(k+1)^2}((k+1)^2\Delta^2 a_{k+1})(k\Delta A_k) + \frac{1}{(k+1)^3}((k+1)^3\Delta^3 a_{k+1})A_k,
\end{aligned}$$

$$\begin{aligned}
\Delta((\Delta b_{k+1})\Delta B_k) &= (\Delta b_{k+2})\Delta B_{k+1} - \Delta b_{k+1}\Delta B_k \\
&= (\Delta b_{k+2})\Delta B_{k+1} - (\Delta b_{k+2})\Delta B_k + (\Delta b_{k+2})\Delta B_k - \Delta b_{k+1}\Delta B_k \\
&= (\Delta b_{k+2})\Delta^2 B_k + (\Delta^2 b_{k+1})\Delta B_k \\
&= \frac{1}{k^2(k+2)}((k+2)\Delta b_{k+2})(k^2\Delta^2 B_k) + \frac{1}{k(k+1)^2}((k+1)^2\Delta^2 b_{k+1})(k\Delta B_k),
\end{aligned}$$

$$
\begin{aligned}
\Delta((\Delta^2 b_k)B_k) &= (\Delta^2 b_{k+1})B_{k+1} - (\Delta^2 b_k)B_k \\
&= (\Delta^2 b_{k+1})B_{k+1} - (\Delta^2 b_{k+1})B_k + (\Delta^2 b_{k+1})B_k - (\Delta^2 b_k)B_k \\
&= (\Delta^2 b_{k+1})\Delta b_k + (\Delta^3 b_{k+1})B_k \\
&= \frac{1}{k(k+1)^2}((k+1)^2\Delta^2 b_{k+1})(k\Delta B_k) + \frac{1}{(k+1)^3}((k+1)^3\Delta^3 b_{k+1})B_k,
\end{aligned}
$$

$$
\begin{aligned}
\Delta((\Delta c_{k+1})\Delta H_k) &= (\Delta c_{k+2})\Delta H_{k+1} - \Delta c_{k+1}\Delta H_k \\
&= (\Delta c_{k+2})\Delta H_{k+1} - (\Delta c_{k+2})\Delta H_k + (\Delta c_{k+2})\Delta H_k - \Delta c_{k+1}\Delta H_k \\
&= (\Delta c_{k+2})\Delta^2 H_k + (\Delta^2 c_{k+1})\Delta H_k \\
&= \frac{1}{k^2(k+2)}((k+2)\Delta c_{k+2})(k^2\Delta^2 H_k) + \frac{1}{k(k+1)^2}((k+1)^2\Delta^2 c_{k+1})(k\Delta H_k),
\end{aligned}
$$

$$
\begin{aligned}
\Delta((\Delta^2 c_k)H_k) &= (\Delta^2 c_{k+1})H_{k+1} - (\Delta^2 c_k)H_k \\
&= (\Delta^2 c_{k+1})H_{k+1} - (\Delta^2 c_{k+1})H_k + (\Delta^2 c_{k+1})H_k - (\Delta^2 c_k)H_k \\
&= (\Delta^2 c_{k+1})\Delta H_k + (\Delta^3 c_{k+1})H_k \\
&= \frac{1}{k(k+1)^2}((k+1)^2\Delta^2 c_{k+1})(k\Delta H_k) + \frac{1}{(k+1)^3}((k+1)^3\Delta^3 c_{k+1})H_k,
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2$.

Since $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$, $\{c_k : k \in \mathbb{Z}\}$ satisfy (H3), $(M_k)_{k\in\mathbb{Z}}$, $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(H_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$ satisfy the Marcinkiewicz condition of order two and $\{c_k : k \in \mathbb{Z}\}$ is bounded by Remark 2.1.1, then $\{k^3\Delta^2 T_k : k \in \mathbb{Z}\}$ is bounded.

We observe that

$$
\begin{aligned}
k^3\Delta^3 N_k &= k^3[\Delta^2 N_{k+1} - \Delta^2 N_k] \\
&= k^3[-N_{k+3}(\Delta T_{k+1} - T_{k+2}T_{k+1}) + N_{k+2}(\Delta T_k - T_{k+1}T_k)] \\
&= -k^3 N_{k+3}[\Delta T_{k+1} - T_{k+2}T_{k+1} - N_{k+3}^{-1}N_{k+2}(\Delta T_k - T_{k+1}T_k)] \\
&= -k^3 N_{k+3}[\Delta T_{k+1} - T_{k+2}T_{k+1} - (I + T_{k+2})(\Delta T_k - T_{k+1}T_k)] \\
&= -k^3 N_{k+3}[\Delta^2 T_k - T_{k+2}T_{k+1} + T_{k+1}T_k - T_{k+2}\Delta T_k + T_{k+2}T_{k+1}T_k] \\
&= -k^3 N_{k+3}[\Delta^2 T_k - 2T_{k+2}\Delta T_k - (\Delta T_{k+1})T_k + T_{k+2}T_{k+1}T_k] \\
&= -N_{k+3}[k^3\Delta^2 T_k - \frac{2k}{k+2}Q_{k+2}(k^2\Delta T_k) - \frac{k^2}{(k+1)^2}((k+1)^2\Delta T_{k+1})Q_k \\
&\quad + \frac{k^2}{(k+1)(k+2)}Q_{k+2}Q_{k+1}Q_k] = N_{k+3}P_k
\end{aligned}
$$

where we have set:

$$
\begin{aligned}
P_k = -N_{k+3}[k^3\Delta^2 T_k - \frac{2k}{k+2}Q_{k+2}(k^2\Delta T_k) - \frac{k^2}{(k+1)^2}((k+1)^2\Delta T_{k+1})Q_k \\
+ \frac{k^2}{(k+1)(k+2)}Q_{k+2}Q_{k+1}Q_k]
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2$.

Since $\{Q_k : k \in \mathbb{Z}\}$, $\{k^2\Delta T_k : k \in \mathbb{Z}\}$ and $\{k^3\Delta^2 T_k : k \in \mathbb{Z}\}$ are bounded, then $\{P_k : k \in \mathbb{Z}\}$ is bounded.

Now, we have that

$$k^3\Delta^3 A_k = k^3\Delta^3(AN_k) = A(k^3\Delta^3 N_k) = AN_{k+3}P_k = A_{k+3}P_k,$$

$$k^3\Delta^3 B_k = k^3\Delta^3(BN_k) = B(k^3\Delta^3 N_k) = BN_{k+3}P_k = B_{k+3}P_k,$$

$$
\begin{aligned}
k^3\Delta^3 H_k &= k^3[(k+3)N_{k+3} - 3(k+2)N_{k+2} + 3(k+1)N_{k+1} - kN_k] \\
&= k^3[kN_{k+3} - 3kN_{k+2} + 3kN_{k+1} - kN_k] + 3k^3N_{k+3} - 6k^3N_{k+2} + 3k^3N_{k+1} \\
&= k^4\Delta^3 N_k + 3k^3\Delta N_{k+2} - 3k^3\Delta N_{k+1} \\
&= k^4\Delta^3 N_k + 3k^3\Delta^2 N_{k+1} \\
&= k(k^3\Delta^3 N_k) + \frac{3k^3}{(k+1)^2}((k+1)^2\Delta^2 N_{k+1}) \\
&= kN_{k+3}P_k + \frac{3k^3}{(k+1)^2}N_{k+3}R_{k+1} \\
&= \frac{k}{k+3}H_{k+3}P_k + \frac{3k^3}{(k+1)^2(k+3)}H_{k+3}R_{k+1},
\end{aligned}
$$

$$
\begin{aligned}
k^3\Delta^3 S_k &= \Lambda k^3[(k+3)N_{k+3} - 3(k+2)N_{k+2} + 3(k+1)N_{k+1} - kN_k] \\
&= \Lambda[kN_{k+3}P_k + \frac{3k^3}{(k+1)^2}N_{k+3}R_{k+1}] \\
&= k\Lambda N_{k+3}P_k + \frac{3k^3}{(k+1)^2}\Lambda N_{k+3}R_{k+1} \\
&= \frac{k}{k+3}S_{k+3}P_k + \frac{3k^3}{(k+1)^2(k+3)}S_{k+3}R_{k+1},
\end{aligned}
$$

and

$$
\begin{aligned}
k^3\Delta^3 M_k &= k^3M[(k+3)^2N_{k+3} - 3(k+2)^2M_{k+2} + 3(k+1)^2N_{k+1} - k^2N_k] \\
&= k^5M\Delta^3 N_k + 3(2k+1)k^3MN_{k+3} + 6k^3MN_{k+3} \\
&- 6(2k+1)k^3MN_{k+2} - 6k^3MN_{k+2} + 3(2k+1)k^3MN_{k+1} \\
&= k^5M\Delta^3 N_k + 3(2k+1)k^3M\Delta N_{k+2} - 3(2k+1)k^3M\Delta N_{k+1} - 6k^3M\Delta N_{k+2} \\
&= k^5M\Delta^3 N_k + 3(2k+1)k^3M\Delta^2 N_{k+1} - 6k^3M\Delta N_{k+2} \\
&= k^2M(k^3\Delta^3 N_k) + \frac{3(2k+1)k^3}{(k+1)^2}[(k+1)\Delta^2 N_{k+1}] - \frac{6k^3}{k+2}[(k+2)\Delta N_{k+2}] \\
&= k^2MN_{k+3}P_k + \frac{3(2k+1)k^3}{(k+1)^2}MN_{k+3}R_{k+1} - \frac{6k^3}{k+2}MN_{k+3}Q_{k+2} \\
&= \frac{k^2}{(k+3)^2}M_{k+3}P_k + \frac{3(2k+1)k^3}{(k+1)^2(k+3)}M_{k+3}R_{k+1} - \frac{6k^3}{(k+2)(k+3)^2}M_{k+3}Q_{k+2}
\end{aligned}
$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2, -3$.

Since $\{A_k : k \in \mathbb{Z}\}$, $\{B_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, $\{H_k : k \in \mathbb{Z}\}$, $\{M_k : k \in \mathbb{Z}\}$, $\{Q_k : k \in \mathbb{Z}\}$, $\{R_k : k \in \mathbb{Z}\}$, and $\{P_k : k \in \mathbb{Z}\}$ are bounded, it follows that $\{k^3\Delta^3 A_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 B_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 H_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 S_k : k \in \mathbb{Z}\}$ and $\{k^3\Delta^3 M_k : k \in \mathbb{Z}\}$ are bounded. This completes the proof.

□

The following remark is in order concerning independence on the parameters regarding the results in this Chapter.

Remark 2.2.6.

- In Theorem 2.2.1, if the problem is well-posed for *some* $p \in (1, \infty)$ then it well-posed for *all* $p \in (1, \infty)$.
- Likewise, In Theorem 2.2.2, Theorem 2.2.3, Theorem 2.2.4, and Theorem 2.2.5, if the problem under consideration is well-posed for *one* set of parameters in the range afforded by the corresponding theorem then it is well-posed for *any* set of parameters in that range.

This is a direct consequence of statement (ii) in each of the mentioned theorems.

CHAPTER 3
WELL-POSEDNESS OF THE EQUATION OF TYPE II

In this Chapter we study the equation of type II. We obtain results similar to those obtained in the Chapter 2. We use the same notation as in Chapter 2.

3.1 The General Well-Posedness Result.

In this section, we establish the general maximal regularity result for solutions of the problem

$$(TIIP_1^2)\begin{cases} (Mu)''(t) - (\Lambda u)'(t) - \dfrac{d}{dt}\displaystyle\int_{-\infty}^{t} c(t-s)u(s)ds \\ = \gamma_\infty u(t) + Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)ds \\ \quad +b_\infty Bu(t) + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \;\; 0 \le t \le 2\pi, \\ \Lambda u(0) = \Lambda u(2\pi),\; Mu(0) = Mu(2\pi), \text{ and } (Mu)'(0) = (Mu)'(2\pi) \end{cases}$$

in the vector-valued Lebesgue, Besov, and Triebel-Lizorkin spaces. As in Chapter 2, A, B, Λ and M are closed linear operators in a Banach space X satisfying $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$, $a, b, c \in L^1(\mathbb{R}_+)$, f is an X-valued function defined on $[0,\, 2\pi]$, and γ_∞, b_∞ are constants.

Let a, b, c be complex valued functions and γ_∞, b_∞ be constants. In the same way as in Chapter 2, define the (M, Λ)-resolvent set of A and B by

$$\rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B) := \{\lambda \in \mathbb{C} :$$
$$\lambda^2 M - (1+\tilde{a}(\lambda))A - (b_\infty + \tilde{b}(\lambda))B - \lambda\Lambda\lambda\tilde{c}(\lambda)I - \gamma_\infty I :$$
$$D(A)\cap D(B) \to X \text{ is bijective and}$$
$$[\lambda^2 M - (1+\tilde{a}(\lambda))A - (b_\infty + \tilde{b}(\lambda))B - \lambda\Lambda - \lambda\tilde{c}(\lambda)I - \gamma_\infty I]^{-1} \in \mathscr{L}(X)\}$$

So, $\lambda \in \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ if and only if the operator

$$[\lambda^2 M - (1+\tilde{a}(\lambda))A - (b_\infty + \tilde{b}(\lambda))B - \lambda\Lambda - \lambda\tilde{c}(\lambda)I - \gamma_\infty I]^{-1}$$

is a linear continuous isomorphism from X onto $D(A)\cap D(B)$. Here we consider $D(A)$, $D(B)$, $D(\Lambda)$ and $D(M)$ as normed spaces equipped with their respective graph norms. All are Banach spaces since the operators are closed.

Using (2.1) we may rewrite (2) in the following way

$$(Mu)''(t) - (\Lambda u)'(t) - \tfrac{d}{dt}(c*u)(t)$$
$$= \gamma_\infty u(t) + Au(t) + (a*Au)(t) + b_\infty Bu(t) + (b*Bu)(t) + f(t),\ 0 \le t \le 2\pi.$$

We now give the definition of solutions of $(TIIP_1^2)$ in our relevant cases.

Definition 3.1.1. *A function $u \in \mathscr{Y}$ is called a strong $\mathscr{Y}$-solution of $(TIIP_1^2)$ if $u \in \mathscr{Y}(D(A))\cap\mathscr{Y}(D(B))$, Λu, Mu, $(Mu)' \in \mathscr{Y}_{per}^{[1]}$, and equation (2) holds for almost all $t \in [0,2\pi]$.*

The counterpart of Lemma 2.1.3 is the following.

Lemma 3.1.2. *Let X be a Banach space, let A, B, Λ, M be closed linear operators in X such that $D(A)\cap D(B) \subset D(\Lambda)\cap D(M)$. Suppose that $a,b,c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ constants and a_k, b_k, c_k are defined by (2.2) and u is a strong $\mathscr{Y}$-solution of*

$(TIIP_1^2)$. *Then*

$$[-k^2 M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_k I - \gamma_\infty I]\hat{u}(k) = \hat{f}(k).$$

for all $k \in \mathbb{Z}$.

Proof. Let $k \in \mathbb{Z}$. Since u is a strong $\mathscr{Y}$-solution of $(TIIP_1^2)$, then $u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B))$, Λu, Mu, $(Mu)' \in \mathscr{Y}_{per}^{[1]}$ and

$$\begin{aligned}&(Mu)''(t) - (\Lambda u)'(t) - \tfrac{d}{dt}(c * u)(t)\\ &= \gamma_\infty u(t) + Au(t) + (a * Au)(t) + b_\infty Bu(t) + (b * Bu)(t) + f(t), \text{ for a.e } t \in [0, 2\pi].\end{aligned}$$

Since $u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B))$, then

$$\hat{u}(k) \in D(A) \cap D(B) \text{ and } \widehat{Au}(k) = A\hat{u}(k),\ \hat{B}u(k) = B\hat{u}(k).$$

by Lemma 1.3.7. Since Λu, Mu, $(Mu)' \in \mathscr{Y}_{per}^{[1]}$, then $\widehat{(\Lambda u)'} = ik\Lambda\hat{u}(k)$, $\widehat{(Mu)'} = ikM\hat{u}(k)$, $\widehat{(Mu)''} = -k^2 M\hat{u}(k)$ by (1.4) and Lemma 1.3.7. Since $u \in \mathscr{Y}(D(A)) \subset L^1(0, 2\pi; D(A))$, $u \in \mathscr{Y}(D(B)) \subset L^1(0, 2\pi; D(B))$ and a, b, $c \in L^1(\mathbb{R}_+)$, then $c * u$, $a * Au$, $b * Bu \in L^1(0, 2\pi; X)$, $(c * u)(0) = (c * u)(2\pi)$ by (1.2) and $\widehat{(c * u)}(k) = \tilde{c}(ik)\hat{u}(k)$, $\widehat{(a * Au)}(k) = \tilde{a}(ik)A\hat{u}(k)$, $\widehat{(b * Bu)}(k) = \tilde{b}(ik)B\hat{u}(k)$ by (1.3). Since $\mathscr{Y} \subset L^1(0, 2\pi; X)$, then u, $(\Lambda u)'$, $(Mu)''$ and $f \in L^1(0, 2\pi; X)$. So u, Au, Bu, $a * Au$, $b * Bu$, $(\Lambda u)'$, $(Mu)''$ and f all belong to $L^1(0, 2\pi; X)$. Then $\frac{d}{dt}(c * u)$ must be in $L^1(0, 2\pi; X)$. Therefore $c * u \in W_{per}^{1,1}(0, 2\pi; X)$ and $\widehat{\frac{d}{dt}(c * u)}(k) = ik\tilde{c}(ik)\hat{u}(k)$ by (1.4).

Taking Fourier series on both sides of (2) we obtain that

$$[-k^2 M - (1+a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_k I - \gamma_\infty I]\hat{u}(k) = \hat{f}(k),\ k \in \mathbb{Z}.$$

□

We adopt the following definition of well-posedness.

Definition 3.1.3. *We say that* $(TIIP_1^2)$ *is* $\mathscr{Y}$*-well-posed, if for each* $f \in \mathscr{Y}$*, there exists a unique strong* $\mathscr{Y}$*-solution* u *of* $(TIIP_1^2)$ *which depends continuously on* f *in* *the sense that the operator* $\mathscr{S} : \mathscr{Y} \to \mathscr{Y}$ *defined by* $\mathscr{S}(f) = u$ *where* u *is the unique strong* $\mathscr{Y}$*-solution of* $(TIIP_1^2)$ *is continuous.*

We observe that the mapping $\mathscr{S}$ appearing in the definition is linear.

We can now establish the following characterization of well-posedess of $(TIIP_1^2)$ in terms of Fourier multipliers.

Theorem 3.1.4. *Let* X *be a Banach space, let* A*,* B*,* Λ*,* M *be closed linear operators in* X *such that* $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$*. Suppose that* $a, b, c \in L^1(\mathbb{R}_+)$*,* γ_∞, b_∞ *are given constants and* a_k*,* b_k*,* c_k *are defined by (2.2). Then the following assertions are equivalent.*

(i) $(TIIP_1^2)$ *is* $\mathscr{Y}$*-well-posed.*

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ *and* $(k^2MN_k)_{k\in\mathbb{Z}}$*,* $(AN_k)_{k\in\mathbb{Z}}$*,* $(BN_k)_{k\in\mathbb{Z}}$*,* $(k\Lambda N_k)_{k\in\mathbb{Z}}$*, and* $(N_k)_{k\in\mathbb{Z}}$ *are* $\mathscr{Y}$*-Fourier multipliers, where*

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}.$$

In this case the following maximal regularity property holds: The unique strong $\mathscr{Y}$*-solution* u *is such that* Au*,* Bu*,* $a*Au$*,* $b*Bu$*,* Λu*,* $(\Lambda u)'$*,* $c*u$*,* $\frac{d}{dt}(c*u)$*,* Mu*,* $(Mu)'$ *and* $(Mu)'' \in \mathscr{Y}$ *and there exists a constant* $C > 0$ *independent of* $f \in \mathscr{Y}$ *such that*

$$\|u\|_{\mathscr{Y}} + \|Au\|_{\mathscr{Y}} + \|Bu\|_{\mathscr{Y}} + \|a * Au\|_{\mathscr{Y}} + \|b * Bu\|_{\mathscr{Y}} + \|\Lambda u\|_{\mathscr{Y}} + \|(\Lambda u)'\|_{\mathscr{Y}}$$
$$+\|c * u\|_{\mathscr{Y}} + \|\frac{d}{dt}(c * u)\|_{\mathscr{Y}} + \|Mu\|_{\mathscr{Y}} + \|(Mu)'\|_{\mathscr{Y}} + \|(Mu)''\|_{\mathscr{Y}} \le C\|f\|_{\mathscr{Y}}.$$

Proof. $(i) \Rightarrow (ii)$. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e^{ikt}y$. Then $\hat{f}(k) = y$. By assumption, there exists a unique strong $\mathscr{Y}$-solution u of $(TIIP_1^2)$. By Lemma 3.1.2, we have that for all $k \in \mathbb{Z}$

$$[-k^2M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I]\hat{u}(k) = y.$$

It follows that

$$[-k^2M-(1+a_k)A-(b_\infty+b_k)B-ik\Lambda-ikc_kI-\gamma_\infty I]$$

is surjective for each $k\in\mathbb{Z}$. Next we prove that

$$[-k^2M-(1+a_k)A-(b_\infty+b_k)B-ik\Lambda-ikc_kI-\gamma_\infty I]$$

is injective for each $k\in\mathbb{Z}$. Let $x\in D(A)\cap D(B)$ such that

$$[-k^2M-(1+a_k)A-(b_\infty+b_k)B-ik\Lambda-ikc_kI-\gamma_\infty I]x=0. \tag{3.1}$$

Define $u(t)=e^{ikt}x$ when $t\in[0,2\pi]$, then $\hat{u}(k)=x$ and $\hat{u}(n)=0$ for all $n\in\mathbb{Z}$, $n\neq k$. By (3.1) we have that

$$\begin{aligned}\widehat{(Mu)''}(n)-\widehat{(\Lambda u)'}(n)-\widehat{\frac{d}{dt}(c*u)}(n)&=\gamma_\infty\hat{u}(n)+\widehat{Au}(n)+\widehat{(a*Au)}(n)\\&\quad+b_\infty\widehat{Bu}(n)+\widehat{(b*Bu)}(n),\end{aligned}$$

for all $n\in\mathbb{Z}$. From the uniqueness theorem of Fourier coefficients, we conclude that u satisfies that

$$\begin{aligned}(Mu)''(t)-(\Lambda u)'(t)-\frac{d}{dt}(c*u)(t)&=\gamma_\infty u(t)+Aw(t)+(a*Au)(t)\\&\quad+b_\infty Bu(t)+(b*Bu)(t)\end{aligned}$$

for almost all $t\in[0,2\pi]$. Thus u is a strong $\mathscr{Y}$-solution of $(TIIP_1^2)$ with $f=0$. We obtain $x=0$ by the uniqueness assumption. We have shown that

$$[-k^2M-(1+a_k)A-(b_\infty+b_k)B-ik\Lambda-ikc_kI-\gamma_\infty I]$$

is injective for each $k\in\mathbb{Z}$. Now we show that

$$N_k=[k^2M+(1+a_k)A+(b_\infty+b_k)B+ik\Lambda+ikc_kI+\gamma_\infty I]^{-1}\in\mathscr{L}(X).$$

Let $k \in \mathbb{Z}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to x$. For each $n \in \mathbb{N}$ we define $f_n(t) = e^{ikt}x_n$ and $f(t) = e^{ikt}x$. Then $f_n, f \in \mathscr{Y}$, for every $n \in \mathbb{N}$ and $f_n \to f$ in $\mathscr{Y}$. Since $(TIIP_1^2)$ is $\mathscr{Y}$-well-posed, then for each $f_n, f \in \mathscr{Y}$ there exists a unique strong $\mathscr{Y}$-solution $\mathscr{S}(f_n) = u_n$, $\mathscr{S}(f) = u$. Since $f_n \to f$ in $\mathscr{Y}$, then $u_n \to u$ in $\mathscr{Y}$ by continuity of $\mathscr{S}$. Therefore $\hat{u}_n(k) \to \hat{u}(k)$ by Remark 2.1.5. Since

$$-k^2M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I$$

is bijective we obtain that $\hat{u}_n(k) = -N_kx_n$, $\hat{u}(k) = -N_kx$ by Lemma 3.1.2, then $N_kx_n \to N_kx$. Thus by the Closed Graph Theorem, $N_k \in \mathscr{L}(X)$. Therefore $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$.

Next we show that if $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, and $S_k = k\Lambda N_k$ for $k \in \mathbb{Z}$, then $(M_k)_{k\in\mathbb{Z}}$, $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers. Since $N_k \in \mathscr{L}(X)$ and A, B, Λ, M are closed, then M_k, A_k, B_k and S_k are bounded for all $k \in \mathbb{Z}$. Now let $f \in \mathscr{Y}$, then there exists an strong $\mathscr{Y}$-solution u of $(TIIP_1^2)$. Then $\hat{u}(k) = -N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$ by Lemma 3.1.2. Therefore

$$\hat{u}(k) \in D(A) \cap D(B) \subset D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$. Since A and B are closed, then

$$\widehat{Au}(k) = A\hat{u}(k) = -AN_k\hat{f}(k) = -A_k\hat{f}(k)$$
$$\widehat{Bu}(k) = B\hat{u}(k) = -BN_k\hat{f}(k) = -B_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$ by Lemma 1.3.7. Since Λ, M are closed and Λu, Mu, $(Mu)' \in \mathscr{Y}_{per}^{[1]}$, then

$$\widehat{(\Lambda u)'}(k) = ik\widehat{\Lambda u}(k) = ik\Lambda\hat{u}(k) = -ik\Lambda N_k\hat{f}(k) = -iS_k\hat{f}(k)$$

$$\widehat{(Mu)'}(k) = ik\widehat{(Mu)}(k) = ikM\hat{u}(k)$$

$$\widehat{(Mu)''}(k) = ik\widehat{(Mu)'}(k) = -k^2M\hat{u}(k) = k^2MN_k\hat{f}(k) = M_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$ by (1.4) and Lemma 1.3.7. It follows that $(M_k)_{k\in\mathbb{Z}}$, $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers. Therefore the implication $(i) \Rightarrow (ii)$ is true.

$(ii) \Rightarrow (i)$. By definition of N_k we have the following equality

$$k^2MN_k + (1+a_k)AN_k + (b_\infty + b_k)BN_k + ik\Lambda N_k + ikc_kN_k + \gamma_\infty N_k = -I,\ k \in \mathbb{Z}$$

which implies that

$$kc_kN_k = i[I+(k^2MN_k+(1+a_k)AN_k+(b_\infty+b_k)BN_k+ik\Lambda N_k+\gamma_\infty N_k)],\ k \in \mathbb{Z}. \quad (3.2)$$

Then by Remarks 2.1.1 and 1.3.5 $(kc_kN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier as well. Since $(k^2MN_k)_{k\in\mathbb{Z}}$, $(kc_kN_k)_{k\in\mathbb{Z}}$, and $(k\Lambda N_k)_{k\in\mathbb{Z}}$ are $\mathscr{Y}$-Fourier multipliers, then $(c_kN_k)_{k\in\mathbb{Z}}$, $(ikc_kN_k)_{k\in\mathbb{Z}}$, $(ik\Lambda N_k)_{k\in\mathbb{Z}}$, $(\Lambda N_k)_{k\in\mathbb{Z}}$, $(ikMN_k)_{k\in\mathbb{Z}}$, and $(MN_k)_{k\in\mathbb{Z}}$ are also $\mathscr{Y}$-Fourier multipliers by Remarks 1.3.5 and 1.3.12. Then for all $f \in \mathscr{Y}$, there exist u, v_1, v_2, v_3, v_4, v_5, v_6, v_7 and $v_8 \in \mathscr{Y}$ such that

$$\hat{u}(k) = N_k\hat{f}(k), \quad (3.3)$$

$$\begin{aligned}
\hat{v}_1(k) &= AN_k\hat{f}(k) = A\hat{u}(k) = \widehat{Au}(k),\\
\hat{v}_2(k) &= BN_k\hat{f}(k) = B\hat{u}(k) = \widehat{Bu}(k),\\
\hat{v}_3(k) &= \Lambda N_k\hat{f}(k) = \Lambda\hat{u}(k) = \widehat{\Lambda u}(k),\\
\hat{v}_4(k) &= MN_k\hat{f}(k) = M\hat{u}(k) = \widehat{Mu}(k),\\
\hat{v}_5(k) &= c_kN_k\hat{f}(k) = c_k\hat{u}(k) = \widehat{c*u}(k),
\end{aligned} \quad (3.4)$$

$$\begin{aligned}
\hat{v}_6(k) &= ik\Lambda N_k\hat{f}(k) = ik\Lambda\hat{u}(k) = ik\widehat{\Lambda u}(k),\\
\hat{v}_7(k) &= ikMN_k\hat{f}(k) = ikM\hat{u}(k) = ik\widehat{Mu}(k)\\
\hat{v}_8(k) &= ikc_kN_k\hat{f}(k) = ik\widehat{c*u}(k).
\end{aligned} \quad (3.5)$$

for all $k \in \mathbb{Z}$ by (1.3) and closedness of A, B, Λ, M. Since $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$, then

$$\hat{u}(k) \in D(A) \cap D(B) \subset D(\Lambda) \cap D(M),$$

for all $k \in \mathbb{Z}$ by (3.3). Since A, B, Λ, and M are closed, then

$$u(t) \in D(A) \cap D(B)$$

and $Au(t) = v_1(t)$, $Bu(t) = v_2(t)$, $\Lambda u(t) = v_3(t)$, $Mu(t) = v_4(t)$ and $(c*u)(t) = v_5(t)$ a.e $t \in [0, 2\pi]$ by (3.4) and Lemma 1.3.7 (here we also use the fact that $\mathscr{Y} \subset L^p(0, 2\pi, X)$). Therefore

$$u \in \mathscr{Y}(D(A)) \cap \mathscr{Y}(D(B)),$$

and $c * u$, Λu, $Mu \in \mathscr{Y}$. Now by (3.5), we have that $c * u$, Λu, $Mu \in \mathscr{Y}_{per}^{[1]}$. Since $(k^2 MN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier, then $(-k^2 MN_k)_{k\in\mathbb{Z}}$ is a $\mathscr{Y}$-Fourier multiplier as well. Then there exists $v_9 \in \mathscr{Y}$ such that

$$\hat{v}_9(k) = -k^2 kMN_k\hat{f}(k) = ik(ikM\hat{u}(k)) = ik(ik\widehat{Mu}(k)) = ik\widehat{(Mu)'}(k),\ k \in \mathbb{Z}, \tag{3.6}$$

by (3.4) and (3.5). Then $(Mu)' \in \mathscr{Y}_{per}^{[1]}$. Thus Λu, Mu, $(Mu)' \in \mathscr{Y}_{per}^{[1]}$. Now, since $\hat{u}(k) = N_k\hat{f}(k)$, we have

$$[-k^2M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_kI - \gamma_\infty I](-\hat{u}(k)) = \hat{f}(k),$$

this means that

$$\begin{aligned}\widehat{(Mw)''}(k) - \widehat{(\Lambda w)'}(k) - \widehat{\frac{d}{dt}(c * w)}(k) &= \gamma_\infty\hat{w}(k) + \widehat{Aw}(k) + \widehat{(a * Aw)}(k)\\ &\quad + b_\infty\widehat{Bw}(k) + \widehat{(b * Bw)}(k) + \hat{f}(k),\end{aligned}$$

for all $k \in \mathbb{Z}$ where $w = -u$. From the uniqueness theorem of Fourier coefficients, we conclude that w satisfies

$$(Mw)''(t) - (\Lambda w)'(t) - \frac{d}{dt}(c * w)(t) = \gamma_\infty w(t) + Aw(t) + (a * Aw)(t) \\ + b_\infty Bw(t) + (b * Bw)(t) + f(t)$$

for almost all $t \in [0, 2\pi]$. Thus w is a strong $\mathscr{Y}$-solution of $(TIIP_1^2)$. To prove uniqueness, let u be a strong $\mathscr{Y}$-solution of $(TIIP_1^2)$ with $f = 0$. Then

$$[-k^2 M - (1 + a_k)A - (b_\infty + b_k)B - ik\Lambda - ikc_k I - \gamma_\infty I]\hat{u}(k) = 0$$

for all $k \in \mathbb{Z}$ by Lemma 2.1.3. Since $ik \in \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ for all $k \in \mathbb{Z}$, then $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$. From the uniqueness theorem of Fourier coefficients we have that $u = 0$. Now we show the continuous dependence of u on f. Let $f \in \mathscr{Y}$, then the unique strong $\mathscr{Y}$-solution of $(TIIP_1^2)$, u is such that $\hat{u}(k) = -N_k \hat{f}(k)$ for all $k \in \mathbb{Z}$ by Lemma 3.1.2 and $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$. Since N_k is a $\mathscr{Y}$-Fourier multiplier, then there exists a bounded linear operator $T \in \mathscr{L}(\mathscr{Y}, \mathscr{Y})$ such that $\widehat{Tf}(k) = \hat{u}(k)$ for all $k \in \mathbb{Z}$ by Remark 1.3.12. Then $Tf = u$, so u depends continuously of f.

The last assertion of the theorem is a direct consequence of the fact that Au, Bu, $a * Au$, $b * Bu$, Λu, $(\Lambda u)'$, $c * u$, $\frac{d}{dt}(c * u)$, Mu, $(Mu)'$ and $(Mu)'' \in \mathscr{Y}$ are defined through the operator valued Fourier multipliers $(-AN_k)_{k\in\mathbb{Z}}$, $(-BN_k)_{k\in\mathbb{Z}}$, $(-a_k AN_k)_{k\in\mathbb{Z}}$, $(-b_k BN_k)_{k\in\mathbb{Z}}$, $(-\Lambda N_k)_{k\in\mathbb{Z}}$, $(-k\Lambda N_k)_{k\in\mathbb{Z}}$, $(-c_k N_k)_{k\in\mathbb{Z}}$, $(-kc_k N_k)_{k\in\mathbb{Z}}$, $(-MN_k)_{k\in\mathbb{Z}}$, $(kMN_k)_{k\in\mathbb{Z}}$ and $(k^2 MN_k)_{k\in\mathbb{Z}}$ respectively (here we use the Remarks 2.1.1, 1.3.5 and 1.3.12).

□

3.2 Characterization of Maximal Regularity on Periodic Lebesgue, Besov and Triebel-Lizorkin Spaces for the Equation of Type *II*

In this section, we give concrete conditions that allow us to apply Theorem 3.1.4. Specifically we obtain conditions under which the sequences $(k^2 MN_k)_{k\in\mathbb{Z}}$,

$(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are Fourier multipliers in the scale of spaces under consideration by use of the operator-valued multiplier theorems established in [4], [6], [7], [15].

Recall that for a scalar sequence $\{d_k : k \in \mathbb{Z}\}$ we will need the following hypotheses:

(H0): $\{d_k : k \in \mathbb{Z}\}$ is bounded.

(H1): $\{d_k : k \in \mathbb{Z}\}$, $\{k\Delta d_k : k \in \mathbb{Z}\}$ are bounded.

(H2): $\{d_k : k \in \mathbb{Z}\}$, $\{k\Delta d_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 d_k : k \in \mathbb{Z}\}$ are bounded.

(H3): $\{d_k : k \in \mathbb{Z}\}$, $\{k\Delta d_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 d_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 d_k : k \in \mathbb{Z}\}$ are bounded.

Theorem 3.2.1. *Let X be a UMD Banach space, $1 < p < \infty$ and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$, satisfies (H1), $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H1), where a_k, b_k, c_k are defined by (2.2). Then the following assertions are equivalent.*

(i) $(TIIP_1^2)$ is L^p-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are R-bounded, where

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}.$$

Proof. $(i) \Longrightarrow (ii)$ Assume that $(TIIP_1^2)$ is L^p-well-posed. Then by Theorem 3.1.4, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers. The R-boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$ and $\{N_k : k \in \mathbb{Z}\}$ now follows from Proposition 1.4.3.

$(ii) \Longrightarrow (i)$ In view of Theorem 3.1.4, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

For each $k \in \mathbb{Z}$ we define $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, and $S_k = k\Lambda N_k$. These operators are bounded because $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A,B)$. Since $\{kc_k :$

$k \in \mathbb{Z}\}$ satisfy (H0), then $\{kc_kN_k : k \in \mathbb{Z}\}$ is R-bounded by Kahane's contraction principle.

Now we note the following equality,

$$M_k + (1 + a_k)A_k + (b_\infty + b_k)B_k + iS_k + ikc_kN_k + \gamma_\infty N_k = -I.$$

which implies that

$$A_k = -\frac{1}{1 + a_k}[I + M_k + (b_\infty + b_k)B_k + iS_k + ikc_kH_k + \gamma_\infty N_k]$$

for each $k \in \mathbb{Z}$ such that $a_k \neq -1$. Since $\{M_k : k \in \mathbb{Z}\}$, $\{B_k : k \in \mathbb{Z}\}$, $\{kc_kN_k : k \in \mathbb{Z}\}$, $\{S_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are R-bounded, then the sequence $\{A_k : k \in \mathbb{Z}\}$ is also R-bounded by Remark 1.4.5 and Remark 2.1.1.

We observe that

$$\begin{aligned}
&N_{k+1}^{-1}N_k \\
&= [(k+1)^2M + (1 + a_k)A + (b_\infty + b_k)B + i(k+1)\Lambda + i(k+1)c_{k+1}I + \gamma_\infty I]N_k \\
&= [N_k^{-1} + (2k+1)M + \Delta a_kA + \Delta b_kB + i\Delta(kc_k)I + i\Lambda]N_k \\
&= I + (2k+1)MN_k + (\Delta a_k)AN_k + (\Delta b_k)BN_k + i[\Delta(kc_k)]N_k + i\Lambda N_k \\
&= I + \frac{2k+1}{k^2}M_k + (\Delta a_k)A_k + (\Delta b_k)B_k + i[\Delta(kc_k)]N_k + \frac{i}{k}S_k
\end{aligned}$$

for all $k \in \mathbb{Z}$, $k \neq 0$.

If we define

$$T_k = \frac{2k+1}{k^2}M_k + (\Delta a_k)A_k + (\Delta b_k)B_k + i[\Delta(kc_k)]N_k + \frac{i}{k}S_k, \tag{3.7}$$

then $N_{k+1}^{-1}N_k = I + T_k$ for all $k \in \mathbb{Z}$, $k \neq 0$.

Define

$$Q_k = -kT_k = -[\frac{2k+1}{k}M_k + k(\Delta a_k)A_k + k(\Delta b_k)B_k + ik[\Delta(kc_k)]N_k + iS_k].$$

for all $k \in \mathbb{Z}$, $k \neq 0$.

Since $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$ and $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H1), then $\{Q_k : k \in \mathbb{Z}\}$ is R-bounded by Remark 1.4.5.

From this point we proceed as in the proof of Theorem 2.2.1 to obtain that $(A_k)_{k\in\mathbb{Z}}$, $(B_k)_{k\in\mathbb{Z}}$, $(S_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$ and $(M_k)_{k\in\mathbb{Z}}$ are L^p-Fourier multipliers.

□

From the proof of Theorem 3.2.1, we deduce the following result for $B^s_{pq}-$solutions in case X has nontrivial Fourier type.

Theorem 3.2.2. *Let X be a Banach space with nontrivial Fourier type and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$, satisfies (H1), $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H0) and (H1), where a_k, b_k, c_k are defined by (2.2). Then the following assertions are equivalent.*

(i) $(TIIP^2_1)$ is L^p-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are bounded, where

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}.$$

Proof. $(i) \Longrightarrow (ii)$. Assume that $(TIIP^2_1)$ is B^s_{pq}-well-posed. Then by Theorem 3.1.4, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ now follows from Remark 1.3.12.

$(ii) \Longrightarrow (i)$. In view of Theorem 3.1.4, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. By Theorem 1.3.10 the proof follows the same lines as that of the preceding theorem.

□

We now consider the problem of well-posedness in Besov spaces $B^s_{pq}(0, 2\pi, X)$ for arbitrary Banach spaces X. For this, assumptions (H0) and (H1) are no longer sufficient.

For Banach spaces with nontrivial Fourier type, we need the Marcinkiewicz condition of order one:

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k\Delta M_k\|) < \infty, \tag{3.8}$$

see Theorem 1.3.10, which is used in the proof of Theorem 3.2.2.

For arbitrary Banach spaces, a Marcinkiewicz condition of order two is needed:

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\|) < \infty, \tag{3.9}$$

see Theorem 1.3.10.

Our next result uses this condition to obtain maximal regularity for $(TIIP^2_1)$ when X does not necessarily have nontrivial Fourier type.

Theorem 3.2.3. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$ satisfy (H2), $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H2) and (H0), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 \le p$, $q \le \infty$, the following statements are equivalent.*

(i) $(TIIP^2_1)$ is B^s_{pq}-well-posed.

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are bounded, where

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}.$$

Proof. $(i) \implies (ii)$. Assume that $(TIIP^2_1)$ is B^s_{pq}-well-posed. Then by Theorem 3.1.4, $i\mathbb{Z} \subset \rho_{M,\tilde{a},\tilde{b},\tilde{c}}(A, B, \Lambda)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$

are B^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ now follows of Remark 1.3.12.

$(ii) \implies (i)$. In view of Theorem 3.1.4, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are B^s_{pq}-Fourier multipliers. Let $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, and $S_k = k\Lambda N_k$. Since (H2) implies (H1), then the verification of the Marcinkiewicz condition of order one is similar to what was done in the proof of Theorem 3.2.1.

It remains to prove that $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 M_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 A_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 B_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 S_k\| < \infty$, and $\sup_{k\in\mathbb{Z}} \|k^2\Delta^2 N_k\| < \infty$.

We recall from the proof of Theorem 3.2.1 that the family $(T_k)_{k\in\mathbb{Z}}$ is defined through

$$T_k = \frac{2k+1}{k^2}M_k + (\Delta a_k)A_k + k(\Delta b_k)B_k + i[\Delta(kc_k)]N_k + \frac{i}{k}S_k,\ k \neq 0$$

such that $N^{-1}_{k+1}N_k = I + T_k$, $Q_k = -kT_k$, $k\Delta N_k = N_{k+1}Q_k$ for all $k \in \mathbb{Z}$, $k \neq 0$, and $\{kT_k : k \in \mathbb{Z}\}$ is bounded.

We observe that for $k \neq 0$,

$$\Delta T_k = \Delta(\frac{2k+1}{k^2}M_k) + \Delta[(\Delta a_k)A_k] + \Delta[k(\Delta b_k)B_k] + i\Delta([\Delta(kc_k)]N_k) + i\Delta(\frac{1}{k}S_k).)$$

We consider each term separately as in the proof of Theorem 2.2.3 to prove that $\sup_{k\in\mathbb{Z}}\{k^2\|\Delta T_k\|\} < \infty$. Using the same computations made in the proof of Theorem

2.2.3 it only remains to prove that $\sup_{k\in\mathbb{Z}} k^2\Delta([\Delta(kc_k)]N_k) < \infty$. But this follows from

$$\begin{aligned}
\Delta([\Delta(kc_k)]N_k) &= (\Delta[(k+1)c_{k+1}])N_{k+1} - [\Delta(kc_k)]N_k \\
&= (\Delta[(k+1)c_{k+1}])N_{k+1} - (\Delta[(k+1)c_{k+1}])N_k \\
&+ (\Delta[(k+1)c_{k+1}])N_k - [\Delta(kc_k)]N_k \\
&= (\Delta[(k+1)c_{k+1}])\Delta H_k + [\Delta^2(kc_k)]N_k \\
&= \frac{1}{k(k+1)}[(k+1)\Delta[(k+1)c_{k+1}]](k\Delta H_k) + \frac{1}{k^2}[k^2\Delta^2(kc_k)]N_k
\end{aligned}$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1$ and the fact that $\{kc_k : k \in \mathbb{Z}\}$ satisfies (H2) and $(N_k)_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz condition of order one.

We observe from (3.7) that for $k \neq 0, -1$

$$\begin{aligned}
k^2\Delta^2 N_k &= k^2[\Delta N_{k+1} - \Delta N_k)] \\
&= k^2[-N_{k+2}T_{k+1} + N_{k+1}T_k] \\
&= -k^2 N_{k+2}[T_{k+1} - N_{k+2}^{-1}N_{k+1}T_k] \\
&= -k^2 N_{k+2}[T_{k+1} - (I + T_{k+1})T_k] \\
&= -k^2 N_{k+2}[T_{k+1} - T_k - T_{k+1}T_k] \\
&= -k^2 N_{k+2}[\Delta T_k - T_{k+1}T_k] \\
&= -N_{k+2}[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k] \\
&= N_{k+2}R_k
\end{aligned}$$

where we have set $R_k = -[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k]$ for all $k \in \mathbb{Z}$, $k \neq 0, -1$.

Since $\{Q_k : k \in \mathbb{Z}\}$ and $\{k^2\Delta T_k : k \in \mathbb{Z}\}$ are bounded, then $\{R_k : k \in \mathbb{Z}\}$ is bounded as well.

From this point we continue as in the proof of the Theorem 2.2.3 to obtain that $\{k^2\Delta^2 N_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 A_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 B_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 S_k : k \in \mathbb{Z}\}$, and $\{k^2\Delta^2 M_k : k \in \mathbb{Z}\}$ are bounded. This completes the proof.

□

From the proof of the Theorem 3.2.3 and using Theorem 1.3.11, we deduce the following result for $F^s_{pq}-$solutions in the case that $1 < p < \infty$, $1 < q \leq \infty$ and $s > 0$.

Theorem 3.2.4. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$ satisfy (H2), $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H2) and (H0), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 < p < \infty$, $1 < q \leq \infty$, the following assertions are equivalent.*

(i) $(TIIP^2_1)$ *is $F^s_{p,q}$-well-posed.*

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, *and* $\{N_k : k \in \mathbb{Z}\}$ *are bounded, where*

$$N_k = [k^2M + (1 + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \Longrightarrow (ii)$. Follows from Theorem 3.1.4 and Remark 1.3.12.

$(ii) \Longrightarrow (i)$. Follows from Theorem 1.3.11 using the same lines as the proof of the preceding theorem.

□

We now consider the problem of well-posedness in the vector-valued Triebel-Lizorkin spaces $F^s_{pq}(0, 2\pi, X)$ with parameters $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $s > 0$. For this, assumption (H2) is no longer sufficient.

A condition which implies that $(M_k)_{k\in\mathbb{Z}}$ is a Fourier multiplier for the scale F^s_{pq}, $s \in \mathbb{R}$, $1 < p < \infty$, $1 < q \leq \infty$ is the Marcinkiewicz condition of order two which is used in the proof of Theorem 3.2.4.

For $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, a Marcinkiewicz condition of order three is needed:

$$\sup_{k\in\mathbb{Z}} (\|M_k\| + \|k\Delta M_k\| + k^2\|\Delta^2 M_k\| + k^3\|\Delta^3 M_k\|) < \infty. \tag{3.10}$$

Our next result uses this condition to obtain characterization of F^s_{pq}-well-posedness of the Problem $(TIIP^2_1)$.

Theorem 3.2.5. *Let X be a Banach space and let A, B, Λ, M be closed linear operators in X such that $D(A) \cap D(B) \subset D(\Lambda) \cap D(M)$. Suppose that $a, b, c \in L^1(\mathbb{R}_+)$, γ_∞, b_∞ are given constants, $\{a_k : k \in \mathbb{Z}\}$, $\{b_k : k \in \mathbb{Z}\}$ satisfy (H3), $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H3) and (H0), where a_k, b_k, c_k are defined by (2.2). Then for $s > 0$ and $1 \le p < \infty$, $1 \le q \le \infty$, the following assertions are equivalent.*

(i) $(TIIP^2_1)$ *is $F^s_{p,q}$-well-posed.*

(ii) $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, *and* $\{N_k : k \in \mathbb{Z}\}$ *are bounded, where*

$$N_k = [k^2M + (a_\infty + a_k)A + (b_\infty + b_k)B + ik\Lambda + ikc_kI + \gamma_\infty I]^{-1}$$

Proof. $(i) \implies (ii)$. Assume that $(TIIP^2_1)$ is F^s_{pq}-well-posed. Then by Theorem 3.1.4, $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A, B)$ and $(k^2MN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are F^s_{pq}-Fourier multipliers. The boundedness of $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{BN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ follows of Remark 1.3.12.

$(ii) \implies (i)$. In view of Theorem 3.1.4, it suffices to show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, $(BN_k)_{k\in\mathbb{Z}}$, $(k\Lambda N_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are F^s_{pq}-Fourier multipliers. Let $M_k = k^2MN_k$, $A_k = AN_k$, $B_k = BN_k$, $S_k = k\Lambda N_k$, and $S_k = k\Lambda N_k$. Since (H3) implies (H2), then the verification of the Marcinkiewicz condition of order two is similar to what was done in the proof of Theorem 3.2.3.

It remains to prove that $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 M_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 A_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 B_k\| < \infty$, and $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 S_k\| < \infty$, $\sup_{k\in\mathbb{Z}} \|k^3\Delta^3 N_k\| < \infty$.

We recall from the proof of Theorem 3.2.1 that

$$T_k = \frac{2k+1}{k^2} M_k + (\Delta a_k)A_k + (\Delta b_k)B_k + i[\Delta(kc_k)]N_k + i\frac{1}{k}S_k,$$

is such that $N_{k+1}^{-1}N_k = I + T_k$, $k\Delta N_k = N_{k+1}Q_k$ where $Q_k = -kT_k$, $\Delta^2 N_k = -N_{k+2}(\Delta T_k - T_{k+1}T_k)$, $k^2\Delta^2 N_k = N_{k+2}R_k$ where $R_k = -[k^2\Delta T_k - \frac{k}{k+1}Q_{k+1}Q_k]$ for all $k \in \mathbb{Z}$, $k \neq 0, -1$, and the sets $\{Q_k : k \in \mathbb{Z}\}$, $\{k^2\Delta^2 T_k : k \in \mathbb{Z}\}$, $\{R_k : k \in \mathbb{Z}\}$ are bounded. Also we showed in the proof of Theorem 3.2.3 that

$$\begin{aligned}\Delta T_k = &\frac{2k+3}{(k+1)^2}\Delta M_k - \frac{2k^2+4k+1}{k^2(k+1)^2}M_k + i\frac{1}{k+1}\Delta S_k - i\frac{1}{k(k+1)}S_k \\ &+ (\Delta a_{k+1})\Delta A_k + (\Delta^2 a_k)A_k + (\Delta b_{k+1})\Delta B_k + (\Delta^2 b_k)B_k \\ &+ i(\Delta[(k+1)c_{k+1}])\Delta N_k + i[\Delta^2(kc_k)]N_k.\end{aligned}$$

Then

$$\begin{aligned}\Delta^2 T_k = &\Delta(\frac{2k+3}{(k+1)^2}\Delta M_k) - \Delta(\frac{2k^2+4k+1}{k^2(k+1)^2}M_k) + i\Delta(\frac{1}{k+1}\Delta S_k) - i\Delta(\frac{1}{k(k+1)}S_k) \\ &+ \Delta((\Delta a_{k+1})\Delta A_k) + \Delta((\Delta^2 a_k)A_k) + \Delta((\Delta b_{k+1})\Delta B_k) + \Delta((\Delta^2 b_k)B_k) \\ &+ i\Delta((\Delta[(k+1)c_{k+1}])\Delta N_k) + i\Delta([\Delta^2(kc_k)]N_k).\end{aligned}$$

We consider each term separately as in the proof of Theorem 2.2.5 to prove that $\sup_{k\in\mathbb{Z}}\{k^3\|\Delta^2 T_k\|\} < \infty$. Using the same computations made in the proof of Theorem 2.2.5 it only remains to prove that $\sup_{k\in\mathbb{Z}} k^3\Delta((\Delta[(k+1)c_{k+1}])\Delta N_k) < \infty$ and $\sup_{k\in\mathbb{Z}} k^3\Delta((\Delta^2[kc_k)N_k) < \infty$. But this follows from

$$\begin{aligned}\Delta((\Delta[(k+1)c_{k+1}])\Delta N_k) &= (\Delta[(k+2)c_{k+2}])\Delta N_{k+1} - (\Delta[(k+1)c_{k+1}])\Delta N_k \\ &= (\Delta[(k+2)c_{k+2}])\Delta N_{k+1} - (\Delta[(k+2)c_{k+2}])\Delta N_k \\ &+ (\Delta[(k+2)c_{k+2}])\Delta N_k - (\Delta[(k+1)c_{k+1}])\Delta N_k \\ &= (\Delta[(k+2)c_{k+2}])\Delta^2 N_k + (\Delta^2[(k+1)c_{k+1}])\Delta N_k \\ &= \frac{1}{k^2(k+2)}[(k+2)\Delta[(k+2)c_{k+2}]](k\Delta^2 N_k) \\ &+ \frac{1}{k(k+1)^2}[(k+1)^2\Delta^2[(k+1)c_{k+1}]](k\Delta N_k),\end{aligned}$$

$$\begin{aligned}
\Delta((\Delta^2[kc_k)N_k) &= (\Delta^2[(k+1)c_{k+1})N_{k+1} - (\Delta^2[kc_k)N_k \\
&= (\Delta^2[(k+1)c_{k+1})N_{k+1} - (\Delta^2[(k+1)c_{k+1})N_k \\
&+ (\Delta^2[(k+1)c_{k+1})N_k - (\Delta^2[kc_k)N_k \\
&= (\Delta^2[(k+1)c_{k+1}])\Delta N_k + (\Delta^3[(k+1)c_{k+1}])N_k \\
&= \frac{1}{k(k+1)^2}[(k+1)^2\Delta^2[(k+1)c_{k+1}]](k\Delta N_k) \\
&+ \frac{1}{(k+1)^3}[(k+1)^3\Delta^3[(k+1)c_{k+1}]]N_k
\end{aligned}$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2$ and the fact that $\{kc_k : k \in \mathbb{Z}\}$ satisfy (H3) and $(N_k)_{k\in\mathbb{Z}}$ satisfy the Marcinkiewicz condition of order two.

We observe that

$$\begin{aligned}
k^3\Delta^3 N_k &= k^3[\Delta^2 N_{k+1} - \Delta^2 N_k] \\
&= k^3[-N_{k+3}(\Delta T_{k+1} - T_{k+2}T_{k+1}) + N_{k+2}(\Delta T_k - T_{k+1}T_k)] \\
&= -k^3 N_{k+3}[\Delta T_{k+1} - T_{k+2}T_{k+1} - N_{k+3}^{-1}N_{k+2}(\Delta T_k - T_{k+1}T_k)] \\
&= -k^3 N_{k+3}[\Delta T_{k+1} - T_{k+2}T_{k+1} - (I + T_{k+2})(\Delta T_k - T_{k+1}T_k)] \\
&= -k^3 N_{k+3}[\Delta^2 T_k - T_{k+2}T_{k+1} + T_{k+1}T_k - T_{k+2}\Delta T_k + T_{k+2}T_{k+1}T_k] \\
&= -k^3 N_{k+3}[\Delta^2 T_k - 2T_{k+2}\Delta T_k - (\Delta T_{k+1})T_k + T_{k+2}T_{k+1}T_k] \\
&= -N_{k+3}[k^3\Delta^2 T_k - \frac{2k}{k+2}Q_{k+2}(k^2\Delta T_k) - \frac{k^2}{(k+1)^2}((k+1)^2\Delta T_{k+1})Q_k \\
&+ \frac{k^2}{(k+1)(k+2)}Q_{k+2}Q_{k+1}Q_k] \\
&= N_{k+3}P_k
\end{aligned}$$

where we have set:

$$P_k = -N_{k+3}[k^3\Delta^2 T_k - \frac{2k}{k+2}Q_{k+2}(k^2\Delta T_k) - \frac{k^2}{(k+1)^2}((k+1)^2\Delta T_{k+1})Q_k + \frac{k^2}{(k+1)(k+2)}Q_{k+2}Q_{k+1}Q_k]$$

for all $k \in \mathbb{Z}$, $k \neq 0, -1, -2$.

Since $\{Q_k : k \in \mathbb{Z}\}$, $\{k^2\Delta T_k : k \in \mathbb{Z}\}$ and $\{k^3\Delta^2 T_k : k \in \mathbb{Z}\}$ are bounded sets, then $\{P_k : k \in \mathbb{Z}\}$ is bounded.

Beyond this point we follow the same line as in the proof of the Theorem 2.2.5 to obtain that $\{k^3\Delta^3 N_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 A_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 B_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^3 S_k : k \in \mathbb{Z}\}$, and $\{k^3\Delta^3 M_k : k \in \mathbb{Z}\}$ are bounded. This completes the proof.

□

As in Chapter 2 we can observe that the assertions as of Remark 2.2.6 remain valid in the present context.

CHAPTER 4
EXAMPLES AND APPLICATIONS

4.1 Relation to some Previous Works on the Subject

A large number of partial differential equations arising in physics and in applied sciences can be written in the form of equation (1); among them there are some famous examples, such as the pseudo-parabolic equations and the Sobolev type equations. Sobolev type equation has the form

$$\Lambda u' = Au + f, \tag{4.1}$$

or more generally, equations or systems in which spatial derivatives are mixed with the time derivative of highest order. Specifically, Equation 4.1 is called *strongly regular* if $\Lambda^{-1}A$ is continuous, *weakly regular* if Λ is invertible but does not dominate A and *degenerate* if Λ is not invertible. Strongly regular Sobolev type equations are also widely known as pseudo-parabolic. The Sobolev equations are of interest not only for the sake of generalizations but also because they arise naturally in a vast variety of applications (e.g. in acoustics, electromagnetics, viscoelasticity, heat conduction etc., see e.g. [34]).

For the periodic case initially, Arendt and Bu [6] deal with the problem $u'(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$. This problem corresponds to (TIP_1^2) with $M = B = 0$, $\Lambda = -I$, $a = c = 0$, and $\gamma_\infty = 0$. The additional condition of our definition of well posedness is obtained automatically by Remark 2.1.8. In this case their results are equivalent to our results by Remarks 1.3.5 and 1.3.12.

Arendt and Bu [6] also consider the problem $u''(t) = Au(t) + f(t)$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. This problem corresponds to (TIP_1^2) with $M = I$, $\Lambda = B = 0$ $a = c = 0$, and $\gamma_\infty = 0$. Here again the additional condition of our definition of well posedness is obtained automatically by Remark 2.1.8. In this case their result are equivalent to our result by Remarks 1.3.12.

V. Keyantuo and C. Lizama [30] have considered well-posedness of (TIP_1^2) when $B = M = 0$ and Λ a scalar operator. The additional condition of our definition of well posedness is obtained automatically by Remark 2.1.8. Their results can be deduced from ours.

V. Keyantuo and C. Lizama [31] have considered well-posedness of (TIP_1^2) when $M = I$, $B = 0$, $a = c = 0$ and $\Lambda = \rho A$ for some $\rho \in \mathbb{R}\backslash\{0\}$. The additional condition of our definition of well posedness is obtained automatically by Remark 2.1.8. Their results can be deduced from ours.

Bu [12] has considered the well-posedness of (TIP_1^2) when $B = \Lambda = 0$, $a = c = 0$, and $\gamma_\infty = 0$. His results follow from ours. With our definition of well-posedness we do not need the a priori the estimate [12, (2.2)]. Thus, he considers the problem

$$\begin{cases} (Mu')'(t) = Au(t) + f(t), \ \ 0 \le t \le 2\pi, \\ u(0) = u(2\pi), \, (Mu')(0) = (Mu')(2\pi). \end{cases}$$

It follows from Theorem 2.2.1 that this problem is L^p-well-posed if and only if $i\mathbb{Z} \subset \rho_{0,M,\tilde{0},\tilde{0},\tilde{0}}(A, 0) = \rho_M(A)$ and $\{k^2 M N_k : k \in \mathbb{Z}\}$ and $\{kN_k : k \in \mathbb{Z}\}$ are R-bounded, where $N_k = (k^2 M + A)^{-1}$. In a similar way, we deduce the results in $B_{p,q}^s$ and $F_{p,q}^s$ using Therem 2.2.3 and Theorem 2.2.5 respectively.

C. Lizama and R. Ponce [35] have considered the well-posedness of $(TIIP_1^2)$ when $B = M = 0$, $a = c = 0$, and $\gamma_\infty = 0$. With our definition of well-posedness we do not need impose additional conditions in the modified resolvent. Their results can be deduced from ours.

4.2 Uniformly Elliptic Operators on Domains

We introduce some facts on uniformly elliptic operators on domains of $\mathbb{R}^n$ in order to discuss the examples that follow. Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 1$. We consider measurable functions α_{ik}, β_k, γ_k, and α_0 $(1 \leq j, k \leq n)$ on Ω. We assume that the following uniform ellipticity condition holds: The functions α_{kj}, β_k, γ_k, α_0 are bounded on Ω, i.e., α_{kj}, β_k, γ_k, $\alpha_0 \in L^\infty(\Omega, \mathbb{C})$ for all $1 \leq j, k \leq n$ and the principal part is elliptic; i.e., there exists a constant $\eta > 0$ such that

$$\mathrm{Re}(\sum_{j,k=1}^{n} \alpha_{kj}(x)\xi_j\overline{\xi_k}) \geq \eta|\xi|^2 \text{ for all } \xi \in \mathbb{C}^n, \text{ a.e. } x \in \Omega. \tag{4.2}$$

The maximal possible η in (4.2) is called the ellipticity constant of the matrix $(\alpha_{jk})_{1\leq j,k\leq n}$. Then we consider the elliptic operator $L : W^{1,2}_{\mathrm{loc}} \to \mathscr{D}(\Omega)'$ given by

$$Lu = -\sum_{k,j=1}^{n} D_j(\alpha_{kj}D_k u) + \sum_{k=1}^{n}(\beta_k D_k u - D_k(\gamma_k u)) + \alpha_0 u.$$

With the help of bilinear forms we will define various realizations of $L \in L^2(\Omega)$ corresponding to diverse boundary conditions. Let V be a closed subspace of $W^{1,2}(\Omega)$ containing $W^{1,2}_0(\Omega)$. We define the form $\alpha_V : V \times V \to \mathbb{C}$ by

$$\alpha_V(u,v) = \int_\Omega \left[\sum_{k,j=1}^{n} \alpha_{kj} D_k u\overline{(D_j v)} + \sum_{k=1}^{n}(\beta_k \overline{v} D_k u + \gamma_k u\overline{D_k v}) + \alpha_0 u\overline{v}\right] dx.$$

Then, there exists $\omega \in \mathbb{R}$ such that $\alpha_V + \omega$ is a densely defined, accretive, continuous, and closed sesquilinear form on V (see [38, Chapter 4. p. 100-101]). One can then associate with $\alpha_V + \omega$ an operator on $L^2(\Omega)$ (see [38, Chapter 1. p. 13]). This means that we can associate with α_V an operator A_V. Formally, A_V is given by the expression $A_V u = Lu$. The operator $-A_V$ generates a C_0-semigroup T_V on $L^2(\Omega)$ (see [38, Proposition 1.51]). The Laplacian or Laplace operator corresponds to $\alpha_{kj} = \delta_{jk}$ (Kronecker symbol), $1 \leq j, k \leq n$, $\beta_k = \gamma_k = 0$.

The role of the space V is to impose boundary conditions. Although the dependence on V is not apparent in the formal expression of A_V, it must be understood that for two different spaces V and W, the operators A_V and A_W are different as they are subject to two different types of boundary conditions.

We now give some examples of boundary conditions. We will say that we have:

- **Dirichlet boundary conditions** if $V = W_0^{1,2}(\Omega)$;
- **Neumann boundary conditions** if $V = W^{1,2}(\Omega)$;

We consider Dirichlet boundary conditions with Ω bounded and we assume the following additional conditions: there exists $\mu > 0$ such that

$$\sum_{k,j=1}^{n} \alpha_{kj}(x)\xi_k\xi_j \leq \mu|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega,$$

α_{kj} is real-valued with $\alpha_{kj} = \alpha_{jk}$, $\beta_k = \gamma_k = 0$, $\alpha_0 \geq 0$. Then, in this case the semigroup T_V is positive, $\|T_V(t)\|_{\mathscr{L}(L^2(\Omega))} \leq 1$ for all $t \geq 0$, and T_V is given by a kernel $p_V(t,x,y)$ such that there exist constants $C > 0$, $b > 0$, and $\delta > 0$ such that

$$|p_V(t,x,y)| \leq Ct^{-n/2}e^{-\delta t}e^{-\frac{|x-y|^2}{4bt}} \tag{4.3}$$

for every $t > 0$ and a.e. $x, y \in \Omega$, see [38, Theorem 4.2, Corollary 6.14 and Theorem 4.28]. For every $r \in (1,\infty)$, the C_0-semigroup T_V extends to a bounded C_0-semigroup T_r on $L^r(\Omega)$ with $\|T_r(t)\|_{\mathscr{L}(L^r(\Omega))} \leq 1$ for all $t \geq 0$, by [38, Theorem 4.28]. By (4.3) there exist $M_r > 0$, and $\delta_r > 0$ depending only on r such that $\|T_r(t)\|_{\mathscr{L}(L^r(\Omega))} \leq M_r e^{-\delta_r t}$ for all $t > 0$ and $r \in (1,\infty)$. Denote now by $-A_r$ the corresponding generator on $L^r(\Omega)$. If $\lambda \in \mathbb{C}$, Re $\lambda > -\delta$, then $\lambda \in \rho(-A_r)$ and

$$R(\lambda, -A_r)u = \int_0^\infty e^{-\lambda t}T_r(t)udt \text{ for all } u \in L^r(\Omega), \tag{4.4}$$

by [5, Theorem 3.1.7]. Let $r \in (1,\infty)$. The C_0-semigroup T_r extends to a bounded holomorphic semigroup on the sector $\Sigma_{\pi/2}$, see [38, Corollary 7.5]. Here $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ is the sector in the complex plane of angle $\theta \in (0,\pi]$. By [5, Theorem 3.7.11] we have that $\Sigma_\pi \subset \rho(-A_r)$ and $\sup_{\lambda \in \Sigma_{\pi-\varepsilon}} \|\lambda R(\lambda, -A_r)\| < \infty$ for all $\varepsilon > 0$. Denote by $\sigma(A_r)$ the spectrum of the operator A_r on $L^r(\Omega)$. By [38, Theorem

7.10], we have that $\sigma(A_r) = \sigma(A_2) \subset (0, \infty)$ for all $r \in (1, \infty)$. By [3, Section 7.2.6] we have that $\lambda R(\lambda, -A_r)$ is R-bounded for all $\lambda \in \Sigma_{\pi/2+\theta_r}$ with $0 < \theta_r < \pi/2$. Since $\lambda \to R(\lambda, -A_r)$ is analytic on $\Sigma_\pi \cup \{\lambda \in \mathbb{C} : \text{Re } \lambda > -\delta_r\}$. Then $R(\lambda, -A_r)$ is R-bounded on every compact subset of $\Sigma_\pi \cup \{\lambda \in \mathbb{C} : \text{Re } \lambda > -\delta_r\}$ by [23, Proposition 3.10]. By Remark 1.4.5, we have that $R(\lambda, -A_r)$ and $\lambda R(\lambda, -A_r)$ are R-bounded on $\Sigma_{\pi/2+\theta_r} \cup \{\lambda \in \mathbb{C} : \text{Re } \lambda \geq -\delta_r/2\}$.

4.3 Examples

We conclude with some examples using uniformly elliptic operators in $L^r(\Omega)$ and multiplication operator by a real valued function on Ω.

Example 4.3.1.

Let us consider the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(m(x)\dfrac{\partial u(t,x)}{\partial t}) + L\dfrac{\partial u(t,x)}{\partial t} \\ = -Lu(t,x) - \int_{-\infty}^{t} a(t-s)Lu(s,x)ds + f(t,x),\ (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0,\ (t,x) \in [0,2\pi] \times \partial\Omega, \\ u(0,x) = u(2\pi,x),\ m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t},\ x \in \Omega, \end{cases} \tag{4.5}$$

where $f \in L^p(0, 2\pi; L^r(\Omega))$ for $1 < p, r < \infty$, m is a real-valued measurable function on Ω such that $m \in L^\infty(\Omega)$. This is the degenerate wave equation with fading memory. The non-degenerate equation is studied in [1], and the reference list of this paper contains additional works on that topic. Maximal regularity for the damped wave equation in the absence of memory effects has been studied in [16] and [31]. The problem (4.5) can also be considered as a modified version of a problem which is considered in Favini-Yagi [25, Example 6.24 p. 197]. They do not incorporate the delay aspect of the equation. They restrict their study to the Hölder spaces. The authors are considered with the evolutionary problem as well. For periodic boundary conditions, we obtain complete characterization of well-posedness in the three scales of spaces: L^p, B^s_{pq}, and F^s_{pq}.

We can rewrite the problem (4.5) in the form

$$\begin{cases} \dfrac{\partial}{\partial t}(m(x)\dfrac{\partial u(t,x)}{\partial t}) + A_r\dfrac{\partial u(t,x)}{\partial t} \\ = -A_r u(t,x) - \int_{-\infty}^{t} a(t-s)A_r u(s,x)ds + f(t,x),\ (t,x)\in[0,2\pi]\times\Omega, \\ u(0,x) = u(2\pi,x),\ m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t},\ x\in\Omega. \end{cases} \tag{4.6}$$

If we suppose that a_k defined by (2.2) satisfies (H1) and the additional condition $|\text{Im } a_k| < 1$ for all $k \in \mathbb{Z}$, then $\frac{k^2 m(x)}{1+a_k+ik} \notin (0,\infty)$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{-A_r,M,\tilde{a},\tilde{0},\tilde{0}}(-A_r,0)$, where M is the multiplication operator by m. By Remark 2.1.1, we have that there exists $N \in \mathbb{N}$ such that $\frac{k^2 m(x)}{ik+1+a_k} \in \mathbb{C}\setminus\Sigma_{\theta_r} \cup \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq \delta_r/2\}$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$ whit $|k| \geq N$. Then $\{(\frac{k^2}{ik+1+a_k}M - A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N\}$ and $\{\frac{k^2}{ik+1+a_k}M(\frac{k^2}{ik+1+a_k}M - A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N\}$ are R-bounded. Since

$$\begin{aligned} &\frac{k}{ik+1+a_k}A_r(\frac{k^2}{ik+1+a_k}M - A_r)^{-1} \\ &= -\frac{k}{ik+1+a_k}I + \frac{k}{ik+1+a_k}\frac{k^2}{ik+1+a_k}M(\frac{k^2}{ik+1+a_k}M - A_r)^{-1}, \end{aligned}$$

then $\{\frac{k}{ik+1+a_k}A_r(\frac{k^2}{ik+1+a_k}M - A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N\}$ is R-bounded as well by Remark 1.4.5. Since $N_k = \frac{1}{ik+1+a_k}(\frac{k^2}{ik+1+a_k}M - A_r)^{-1}$, then we have shown that $\{k^2 M N_k : k \in \mathbb{Z}, |k| \geq N\}$, $\{kA_r N_k : k \in \mathbb{Z}, |k| \geq N\}$ and $\{kN_k : k \in \mathbb{Z}, |k| \geq N\}$ are R-bounded. Also by Remark 1.4.5, we have that $\{kN_k : k \in \mathbb{Z}\}$, $\{kA_r N_k : k \in \mathbb{Z}\}$ and $\{k^2 M N_k : k \in \mathbb{Z}\}$ are R-bounded. Therefore, by Theorem 2.2.1, it follows that (4.8) is $L^p(0,2\pi; L^r(\Omega))$-well-posed for all $1 < p < \infty$. Since R-boundedness implies uniform boundedness, then if we suppose that $f \in B^s_{pq}(0,2\pi; L^r(\Omega))$ and a_k satisfies (H1) with $|\text{Im } a_k| < 1$ for all $k \in \mathbb{Z}$, then we have that (4.8) is $B^s_{pq}(0,2\pi; L^r(\Omega))$-well-posed for all $s > 0$, $1 \leq p,q \leq \infty$ by Theorem 2.2.2. Observe that here we include the scale of vector-valued Hölder spaces C^s, $0 < s < 1$. In the F^s_{pq} case if $f \in F^s_{pq}(0,2\pi; L^r(\Omega))$ and a_k satisfies (H3) with $|\text{Im } a_k| < 1$ for all $k \in \mathbb{Z}$, then we have that (4.8) is $F^s_{pq}(0,2\pi; L^r(\Omega))$-well-posed for all $s > 0$, $1 \leq p < \infty$,

$1 \leq q \leq \infty$, by Theorem 2.2.5. Observe that if $s > 0$, $1 < p < \infty$ and $1 < q \leq \infty$ we only need the (H2) condition for this scale. As a particular example we have that $a(t) = e^{-\varepsilon t}\frac{t^{\nu-1}}{\Gamma(\nu)}$ with $\varepsilon > 0$ and $\nu > 0$ satisfies the required conditions for a_k in all the cases.

Example 4.3.2.

Let us consider the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(m(x)\dfrac{\partial u(t,x)}{\partial t}) + L\dfrac{\partial u(t,x)}{\partial t} \\ = Lu(t,x) + \int_{-\infty}^{t} a(t-s)Lu(s,x)ds + f(t,x),\ (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0,\ (t,x) \in [0,2\pi] \times \partial\Omega, \\ u(0,x) = u(2\pi,x),\ m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t},\ x \in \Omega, \end{cases} \tag{4.7}$$

where m is a real-valued measurable function on Ω such that $m(x) \geq 0$, and $f \in L^p(0,2\pi; L^r(\Omega))$ for $1 < p, r < \infty$.

Following Example 4.3.1, we can rewrite the problem (4.7) in the form

$$\begin{cases} \dfrac{\partial}{\partial t}(m(x)\dfrac{\partial u(t,x)}{\partial t}) + A_r\dfrac{\partial u(t,x)}{\partial t} \\ = A_r u(t,x) + \int_{-\infty}^{t} a(t-s)A_r u(s,x)ds + f(t,x),\ (t,x) \in [0,2\pi] \times \Omega, \\ u(0,x) = u(2\pi,x),\ m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t},\ x \in \Omega. \end{cases} \tag{4.8}$$

If we suppose that a_k defined by (2.2) satisfies (H1) and the additional condition $\mathrm{Re}\ a_k > -1$ for all $k \in \mathbb{Z}$, then $\mathrm{Re}\ \frac{k^2 m(x)}{1+a_k-ik} \geq 0$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{A_r,M,\tilde{a},\tilde{0},\tilde{0}}(A_r, 0)$ and $\{(\frac{k^2}{1+a_k-ik}M + A_r)^{-1} : k \in \mathbb{Z}\}$, $\{\frac{k^2}{1+a_k-ik}M(\frac{k^2}{1+a_k-ik}M + A_r)^{-1} : k \in \mathbb{Z}\}$ are R-bounded, here M is the multiplication operator by m. By Remarks (1.4.5), we have that $\{\frac{k}{1+a_k-ik}(\frac{k^2}{1+a_k-ik}M + A_r)^{-1} : k \in \mathbb{Z}\}$ is also R-bounded. Since

$$\begin{aligned} &\frac{k}{1+a_k-ik}A_r(\frac{k^2}{1+a_k-ik}M + A_r)^{-1} \\ &= \frac{k}{1+a_k-ik}I - \frac{k}{1+a_k-ik}\frac{k^2}{1+a_k-ik}M(\frac{k^2}{1+a_k-ik}M + A_r)^{-1}, \end{aligned}$$

then $\{\frac{k}{1+a_k-ik}A_r(\frac{k^2}{1+a_k-ik}M+A_r)^{-1}: k\in\mathbb{Z}\}$ is R-bounded as well by Remark (1.4.5). Since $N_k=\frac{1}{1+a_k-ik}(\frac{k^2}{1+a_k-ik}M+A_r)^{-1}$, then we have show that $\{k^2MN_k : k\in\mathbb{Z}\}$, $\{kA_rN_k : k\in\mathbb{Z}\}$ and $\{kN_k : k\in\mathbb{Z}\}$ are R-bounded. Therefore, by Theorem 2.2.1, we have that (4.8) is $L^p(0,2\pi;L^r(\Omega))$-well-posed for all $1<p<\infty$. Since R-boundedness implies uniform boundedness, then if we suppose that $f\in B^s_{pq}(0,2\pi;L^r(\Omega))$ and a_k satisfies (H1) with Re $a_k>-1$ for all $k\in\mathbb{Z}$, then we have that (4.8) is $B^s_{pq}(0,2\pi;L^r(\Omega))$-well-posed for all $s>0$, $1\le p,q\le\infty$ by Theorem 2.2.2. Observe that here we include the scale of vector-valued Hölder spaces C^s, $0<s<1$. In the F^s_{pq} case if $f\in F^s_{pq}(0,2\pi;L^r(\Omega))$ and a_k satisfies (H3) with Re $a_k>-1$ for all $k\in\mathbb{Z}$, then we have that (4.8) is $F^s_{pq}(0,2\pi;L^r(\Omega))$-well-posed for all $s>0$, $1\le p<\infty$, $1\le q\le\infty$, by Theorem 2.2.5. Observe that if $s>0$, $1<p<\infty$ and $1<q\le\infty$ we only need the (H2) condition for this scale. As in the Example 4.3.1, a particular example of $a(t)$ we have $a(t)=e^{-\varepsilon\frac{t^{\nu-1}}{\Gamma(\nu)}}$ with $\varepsilon>0$ and $\nu>0$ that satisfies the required conditions for a_k in all cases.

Example 4.3.3.

Let us consider the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(m_2(x)\dfrac{\partial u(t,x)}{\partial t})-m_1(x)\dfrac{\partial u(t,x)}{\partial t} \\ =Lu(t,x)+\int_{-\infty}^t a(t-s)Lu(s,x)ds+f(t,x),\ (t,x)\in[0,2\pi]\times\Omega, \\ u(t,x)=\frac{\partial u(t,x)}{\partial t}=0,\ (t,x)\in[0,2\pi]\times\partial\Omega, \\ u(0,x)=u(2\pi,x),\ m_2(x)\frac{\partial u(0,x)}{\partial t}=m_2(x)\frac{\partial u(2\pi,x)}{\partial t},\ x\in\Omega, \end{cases} \tag{4.9}$$

where m_1 and m_2 are real-valued measurable functions on Ω such that $m_2(x)\ge 0$ and $\tau<|m_1(x)|\le\mu$ for some τ, $\mu>0$, and $f\in L^p(0,2\pi;L^r(\Omega))$ for $1<p,r<\infty$.

Following the Example 4.3.1, we can rewrite the problem (4.9) in the form

$$\begin{cases} \dfrac{\partial}{\partial t}(m_2(x)\dfrac{\partial u(t,x)}{\partial t})-m_1\dfrac{\partial u(t,x)}{\partial t} \\ =A_ru(t,x)+\int_{-\infty}^t a(t-s)A_ru(s,x)ds+f(t,x),\ (t,x)\in[0,2\pi]\times\Omega, \\ u(0,x)=u(2\pi,x),\ m_2(x)\frac{\partial u(0,x)}{\partial t}=m_2(x)\frac{\partial u(2\pi,x)}{\partial t},\ x\in\Omega. \end{cases} \tag{4.10}$$

If we suppose that Re $a_k > -1$ for all $k \in \mathbb{Z}$, then $\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1} \notin (-\infty, 0)$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{0},\tilde{0}}(A_r, 0)$, where Λ and M are the multiplication operators by m_1 and m_2 respectively. By Remark 2.1.1, we have that there exists $N \in \mathbb{N}$ such that $\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1} \in \Sigma_{\pi/2+\theta_r}$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$ whit $|k| \geq N$. Then $\{\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ is R-bounded. Since $\{\frac{1}{km_2(x)+im_1(x)} : k \in \mathbb{Z}, x \in \Omega\}$ is bounded, then $\{\frac{k}{a_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded by Remark 1.4.5. Since m_1 is bounded then, by Remark 1.4.5 we have that $\{\frac{km_1(x)}{a_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded. Therefore, $\{\frac{k^2 m_2(x)}{a_k+1}(\frac{k^2 m_2(x)+ikm_1(x)}{a_k+1} + A_r)^{-1} : k \in \mathbb{Z}, |k| \geq N, x \in \Omega\}$ are R-bounded. Since $N_k = \frac{1}{a_k+1}(\frac{k^2}{a_k+1}M + i\frac{k}{a_k+1}\Lambda + A_r)^{-1}$, then we have shown that $\{kN_k : k \in \mathbb{Z}, |k| \geq N\}$, $\{k\Lambda N_k : k \in \mathbb{Z}, |k| \geq N\}$ and $\{k^2 MN_k : k \in \mathbb{Z}, |k| \geq N\}$ are R-bounded. By Remark 1.4.5, we have that $\{kN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$ and $\{k^2 MN_k : k \in \mathbb{Z}\}$ are R-bounded. Under the same conditions over a_k in the Example 4.3.2 and $f \in \mathscr{Y}$ we can apply Theorems 2.2.1, 2.2.2, 2.2.4 and 2.2.5 to obtain that the (4.10) is $\mathscr{Y}$-well-posed.

Example 4.3.4.

Let us consider the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial t}(m_2(x)\dfrac{\partial u(t,x)}{\partial t}) - m_1(x)\dfrac{\partial u(t,x)}{\partial t} - \dfrac{\partial}{\partial t}\displaystyle\int_{-\infty}^{t} c(t-s)u(s,x)ds \\ = Lu(t,x) + \int_{-\infty}^{t} a(t-s)Lu(s,x)ds + \int_{-\infty}^{t} b(t-s)m_0(x)u(s,x)ds \\ +f(t,x),\ (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0,\ (t,x) \in [0,2\pi] \times \partial\Omega, \\ u(0,x) = u(2\pi,x),\ m_2(x)\frac{\partial u(0,x)}{\partial t} = m_2(x)\frac{\partial u(2\pi,x)}{\partial t},\ x \in \Omega, \end{cases} \tag{4.11}$$

where m_0, m_1, and m_2 are real-valued measurable functions on Ω such that $0 \leq m_0(x) \leq \mu$, $\tau < |m_1(x)| \leq \mu$, $0 \leq m_2(x)$, for some $\mu, \tau > 0$, all $x \in \Omega$, and $f \in L^p(0, 2\pi; L^r(\Omega))$ for $1 < p, r < \infty$.

Following the Example 4.3.1, we can rewrite the problem (4.11) in the form

$$\begin{cases} \frac{\partial}{\partial t}(m_2(x)\frac{\partial u(t,x)}{\partial t}) - m_1(x)\frac{\partial u(t,x)}{\partial t} - \frac{\partial}{\partial t}\int_{-\infty}^{t} c(t-s)u(s,x)ds \\ = A_r u(t,x) + \int_{-\infty}^{t} a(t-s)A_r u(s,x)ds + \int_{-\infty}^{t} b(t-s)m_0(x)u(s,x)ds \\ +f(t,x),\ (t,x) \in [0,2\pi]\times\Omega, \\ u(0,x) = u(2\pi,x),\ m_2(x)\frac{\partial u(0,x)}{\partial t} = m_2(x)\frac{\partial u(2\pi,x)}{\partial t},\ x \in \Omega. \end{cases} \tag{4.12}$$

If we suppose that Re $a_k > -1$, Re $b_k \geq 0$, and kIm $c_k \leq 0$ for all $k \in \mathbb{Z}$, then $\frac{k^2 m_2(x)+ikm_1(x)+b_k m_0(x)+ikc_k}{a_k+1} \notin (-\infty,0)$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$. Therefore $i\mathbb{Z} \subset \rho_{\Lambda,M,\tilde{a},\tilde{b},\tilde{c}}(A_r,B)$, where B, Λ, and M are the multiplication operators by m_0, m_1, and m_2 respectively. In the similar way that in the Example 4.3.3 we can show that $\{kN_k : k \in \mathbb{Z}\}$, $\{kBN_k : k \in \mathbb{Z}\}$, $\{k\Lambda N_k : k \in \mathbb{Z}\}$ and $\{k^2 MN_k : k \in \mathbb{Z}\}$ are R-bounded where $N_k = \frac{1}{a_k+1}(\frac{k^2}{a_k+1}M + i\frac{k}{a_k+1}\Lambda + \frac{b_k}{a_k+1}B + i\frac{kc_k}{a_k+1}I + A_r)^{-1}$. Examples of function a that satisfied the required propertied to apply Theorems 2.2.1, 2.2.2, 2.2.4 and 2.2.5 is given in Example 4.3.2, as example for b and c we can take the function $e^{-\varepsilon t}$, $\varepsilon > 0$. With $f \in \mathscr{Y}$ and the appropriate a, b, and c we can obtain that the (4.12) is $\mathscr{Y}$-well-posed.

Example 4.3.5.

Let us consider the boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}(m(x)\frac{\partial u(t,x)}{\partial t}) = A_r u(t,x) + \int_{-\infty}^{t} a(t-s)A_r u(s,x)ds + f(t,x),\ (t,x) \in [0,2\pi]\times\Omega, \\ u(t,x) = \frac{\partial u(t,x)}{\partial t} = 0,\ (t,x) \in [0,2\pi]\times\partial\Omega, \\ u(0,x) = u(2\pi,x),\ m(x)\frac{\partial u(0,x)}{\partial t} = m(x)\frac{\partial u(2\pi,x)}{\partial t},\ x \in \Omega, \end{cases} \tag{4.13}$$

where $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 measure subset of $\mathbb{R}^n$ with positive measure, $m(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ 0 & \text{if } x \in \Omega_2, \end{cases}$ $f \in L^p(0,2\pi; L^r(\Omega))$ for $1 < p, r < \infty$ and $a \in L^1(\mathbb{R}_+)$ such that Re $a_k > -1$ where a_k is define in (2.2). Since Re $\frac{k^2 m(x)}{1+a_k} > 0$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$, then $i\mathbb{Z} \subset \rho_{0,M,\tilde{a},\tilde{0},\tilde{0}}(A_r,0)$ and $\{(\frac{k^2}{1+a_k}M + A_r)^{-1} : k \in \mathbb{Z}\}$,

$\{\frac{k^2}{1+a_k}M(\frac{k^2}{1+a_k}M+A_r)^{-1} : k \in \mathbb{Z}\}$ are R-bounded, here M is the multiplication operator by m. The set $\{\frac{k}{1+a_k}(\frac{k^2}{a_k+1}M + A_r)^{-1}$ is not bounded. Since $N_k = \frac{1}{1+a_k}(\frac{k^2}{1+a_k}M + A_r)^{-1}$, then we have that $\{k^2 M N_k : k \in \mathbb{Z}\}$ is R- bounded, but $\{kN_k : k \in \mathbb{Z}\}$ are not R-bounded. So the problem (4.13) is not $L^p(0, 2\pi; L^r(\Omega))$-well-posed. But nevertheless the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}(m(x)u(t,x)) = A_r u(t,x) + \displaystyle\int_{-\infty}^{t} a(t-s)A_r u(s,x)ds + f(t,x),\ (t,x) \in [0,2\pi] \times \Omega, \\ u(t,x) = 0,\ (t,x) \in [0,2\pi] \times \partial\Omega, \\ m(x)u(0,x) = m(x)u(2\pi,x),\ \frac{\partial}{\partial t}(m(x)u(0,x)) = \frac{\partial}{\partial t}(m(x)u(2\pi,x)),\ x \in \Omega, \end{cases} \tag{4.14}$$

where $\Omega = \Omega_1 \cup \Omega_2$ with Ω_1 and Ω_2 measure subset of $\mathbb{R}^n$ with positive measure, $m(x) = \begin{cases} 1 & \text{if } x \in \Omega_1, \\ 0 & \text{if } x \in \Omega_2, \end{cases}$ $f \in L^p(0, 2\pi; L^r(\Omega))$ for $1 < p, r < \infty$ and $a \in L^1(\mathbb{R}_+)$ such that Re $a_k > -1$ where a_k is define in (2.2) it is. In fact, since Re $\frac{k^2 m(x)}{1+a_k} > 0$ for all $x \in \Omega$ and all $k \in \mathbb{Z}$, then $i\mathbb{Z} \subset \rho_{0,M,\tilde{a},\tilde{0},\tilde{0}}(A_r, 0)$ and $\{(\frac{k^2}{1+a_k}M + A_r)^{-1} : k \in \mathbb{Z}\}$, $\{\frac{k^2}{1+a_k}M(\frac{k^2}{1+a_k}M + A_r)^{-1} : k \in \mathbb{Z}\}$ are R-bounded, here M is the multiplication operator by m. Since $N_k = \frac{1}{1+a_k}(\frac{k^2}{1+a_k}M + A_r)^{-1}$, then we have that $\{k^2 M N_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are R-bounded. Therefore, by Theorem 2.2.1, we have that (4.8) is $L^p(0, 2\pi; L^r(\Omega))$-well-posed for all $1 < p < \infty$.

Remark 4.3.6.

In the case of Neumann boundary conditions, the operator A_r is not invertible. In order to apply the results to this case, we can add in the right side of each of the above equations the term $\eta u(t,x)$ for some $\eta > 0$. Then the above conclusions hold in this case as well.

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