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CLASSIFICATION OF SPLITTING INTERVAL ALGEBRAS AND THE C* EXPONENTIAL LENGTH

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ABSTRACT

Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ be a C^* algebra where $A_n = \bigoplus_{i=1}^{k_n} A_n^i$, A_n^i are splitting interval algebras. Suppose that A has the ideal property: each closed two-sided ideal is generated by the projections inside the ideal, as a closed two-sided ideal. In Chapter 1, we show that the scaled ordered K_0 group and the ordered vector spaces $\mathrm{AffT}(eAe)$ with maps between $\mathrm{AffT}(e'Ae')$ and $\mathrm{AffT}(eAe)$ are the complete invariant for the classification of this class of C^*-algebras, where $eAe := \{eae|\ a \in A\}$, and e, e' are certain projections in A with $e' < e$. We call this invariant Stevens' Invariant.

In Chapter 2, we enlarge the Elliott's invariant by considering the infinite traces. And we show that if A and B have isomorphic Stevens' Invariant, then they have isomorphic Elliott Invariant and vise versa, where A and B are two C^*-algebras with the ideal property. Moreover, for $\mathcal{Z}$-absorbing C^*-algebra, we give a characterization of Cuntz comparability by lower semi-continuous dimension functions.

In Chapter 3, we talk about the C^* exponential length. Let X be a compact Hausdorff space. We give an example to show that there is $u \in \mathrm{C}(X) \otimes \mathrm{M}_n$ with $\det(u(x)) = 1$ for all $x \in X$ and $u \sim_h 1$ such that the C^* exponential length of u (denoted by $cel(u)$) can not be controlled by π. Moreover, for any $\varepsilon > 0$, we can find a simple inductive limit C^*-algebra (simple AH algebra), say A, and a unitary $u \in CU(A)$ with $u \sim_h 1$ and $cel(u) \geq 2\pi - \varepsilon$.

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Contents

Introduction

An AI algebra (approximate interval algebra) is a separable C^*-algebra which is the inductive limit of a sequence of finite direct sums of matrix algebras over $C[0,1]$ (i.e., $A = \varinjlim(A_n, \phi_{n,m})$, where $A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C[0,1])$, with k_n and $[n,i]$ integers). In 1991, George Elliott proved that for simple AI algebras, the pair of functors (K_0, T) is a complete invariant for isomorphism. In other words, $A \cong B$ is equivalent to $(K_0(A), T(A)) \cong (K_0(B), T(B))$.

Since then, successful classification results have been obtained for AH algebras—the interval replaced by more general compact metric space assumed to have dimension at most three—in two important cases: the classification of real rank zero AH algebras (see [11], [19] [20], [16], [17], [5], [22], [23], [7], [10] and [6]) and the classification of simple AH algebras (see [14], [15], [34], [24] and [18]).

The ideal property (each closed two-sided ideal is generated by its projections) unifies and generalizes the above two cases. In 1995, Kenneth H. Stevens proved that unital approximately divisible AI algebras with the ideal property can be classified by their K_0 groups and tracial state spaces (see [59]). Then C. Pasnicu studied C^*-algebras with the ideal property and obtained a characterization theorem for AH algebras with the ideal property.

As pointed out in [25], there are natural examples arising from the C^*-dynamical systems which have the ideal property but are neither of real rank zero nor simple. Therefore it is important to classify C^*-algebras with the ideal property, which may have applications even to the case of real rank zero C^*-algebras (see [25]). Recently, Kui Ji and Chunlan Jiang improved K. H. Stevens's result ([30]). They completely classified all AI algebras with the ideal property. Their proof is quite different from that of Stevens.

In the first chapter of this paper, we generalize the result of Ji and Jiang to the case of limits of splitting interval algebras with the ideal property. We use the so called Stevens' Invariant to classify all the limits of splitting interval algebras with the ideal property. It's a joint work of me with Professor Chunlan Jiang. A splitting interval algebra is of the following form

$$\mathscr{S}(\bar{n}_0, \bar{n}_1) = \{f \in M_n(C[0,1]) : \ f(x) \in \bigoplus_{i=1}^{r_x} M_{n_{x_i}}(\mathbb{C}), \quad x = 0, 1\},$$

where each $\bar{n}_x = (n_{x_1}, \ \cdots, \ n_{x_{r_x}})$, for $x = 0$ or 1, is a partition of n (by positive integers). Let us write $f(x) = \bigoplus_{i=1}^{r_x} f(x_i)$, where $x = 0, 1$, and call 0_i, 1_i fractional points.

An AI algebra can be seen as a special case of a splitting interval algebra, for which there is only one block at the endpoints 0 and 1. Hongbing Su and Xinhui Jiang have classified the real rank zero case and the simple case for splitting interval algebras (see [60] and [31]).

The first chapter consists of four sections. Section 1.1 reviews some definitions and basic results. Section 1.2 and Section 1.3 discusses the Existence Theorem and Uniqueness Theorem, respectively. In Section 1.4, we use the Existence Theorem, Uniqueness Theorem and the dichotomy theorem to obtain our classification theorem.

In Chapter 2, we showed that the Stevens' invariant is equivalent to the Elliott's invariant for the C*-algebras with the ideal property when infinity value traces are considered. For the traditional Elliott's invariant, infinite traces are not included. We enlarge the Elliott's invariant by considering the infinite traces and showed that the enlarged Elliott's invariant is equivalent to the Stevens' invariant. This result strengthens the power of the Elliott's invariant and shows that Elliott's invariant may tell us more information of C*-algebras.

Recent examples due first to Rørdam and later Toms have shown the currently proposed invariants to be insufficient for the classification of all simple, separable, and nuclear C*-algebras. That is, there are simple, separable, and nuclear C*-algebras that can be distinguished by their Cuntz semigroups but not by their Elliott's Invariant. So the Cuntz semigroup has become popular and important. In Chapter 2, we also give a characterization of Cuntz comparability by lower semi-continuous dimension functions for C*-algebras whose Cuntz semigroup is almost unperforated. This result is based on a result of Rørdom.

In chapter 3, we talk about the C* exponential length. Exponential rank was introduced by Phillips and Ringrose [53]; and, subsequently, exponential length was introduced by Ringrose [55]. These invariants have been fundamental in the structure and classification of C*-algebras. Among other things, they have played important roles in factorization and approximation properties for C*-algebras e.g., the weak FU property [46], Weyl-von Neumann Theorems [39], [40] (which in turn have been important in various generalizations of BDF Theory beyond the Calkin algebra case), and the uniqueness theorems of classification theory [18], [41]. The C* exponential length and rank have been extensively studied (see [55], [53], [46], [51], [66], [68], [67], [26], [50], [48], [36], [52], [62], [37], etc. -an incomplete list).

In [50], N. C. Phillips calculates the exponential rank of simple C*-algebra B with representation $B = \lim_{\rightarrow} B_i$, where $B_i = \bigoplus_{t=1}^{s(i)} C(X_{it}) \bigotimes M_{n(i,t)}$, and X_{it} are compact metric spaces such that $\sup_{i,t} \dim(X_{it}) < \infty$. He also studies the exponential length for $u \in \bigcup_{i=1}^{\infty} B_i$. In particular, he mentions that (see [50] page 851 the paragraphs before Proposition 7.9), "We believe that suitable modifications of Lemma 5.2 and 5.3 will show that if $u \sim_h 1$ and $\det(u) = 1$, then $cel(u) \leq \pi$ (even though, for general u, $cel(u)$ can be arbitrarily large)." However, in this paper we provide a method for constructing counterexamples to this conjecture. In fact, for any $\varepsilon > 0$, we can find a simple inductive limit C*-algebra (simple AH algebra), say A, and a unitary $u \in CU(A)$ with $u \sim_h 1$ and $cel(u) \geq 2\pi - \varepsilon$ (see Corollary 3.2.16 and Theorem 3.2.17). Note that for unital real rank zero C*-algebras, π is an upper bound for the C* exponential length (see [36]). In the process of our proof, we show that there are unitaries $u_i \in A_i$ with $u_i \sim_h 1$ and $\det(u_i) = 1$ whose C* exponential length are close to 2π (see Theorem 3.2.15 and the proof of 3.2.16). At last, we conclude that for the C*-algebra A we constructed, $cel_{CU}(A) = 2\pi$ (see Theorem 3.2.18).

In a recent paper of H. Lin (see [37]), he gets a similar example of the Theorem 3.2.12 in our paper with different method. He also provides examples compare to our Theorem 3.2.17 with lower bound π but not 2π (see 5.11, 5.12 of [37]). Therefore, by our result, 2π is an optimal bound of $cel_{CU}(A)$ for unital separable simple C*-algebra A with tracial rank less and equal than 1 (c.f. Lemma 4.5 of [37]).

Chapter 1

A complete classification of limits of splitting interval algebras with the ideal property

Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ be an inductive limit C*-algebra with $A_n = \bigoplus_{i=1}^{k_n} A_n^i$ where A_n^i are splitting interval algebras. Suppose that A has the ideal property: each closed two-sided ideal is generated (as an ideal) by projections in the ideal. In this chaptr, we show that the scaled ordered K_0 group and the ordered vector spaces $\mathrm{AffT}(eAe)$ (where e is a projection) together with the natural maps between them are complete invariants for the classification of this class of C*-algebras.

1.1 Notation and Preliminaries

Definition 1.1.1. *A splitting interval algebra is any C*-algebra of the form*

$$\mathscr{S}(\bar{n}_0, \bar{n}_1) = \{f \in M_n(C[0,1]) : \; f(x) \in \bigoplus_{i=1}^{r_x} M_{n_{x_i}}(\mathbb{C}), \quad x = 0, 1\},$$

where each $\bar{n}_x = (n_{x_1}, \;\cdots, \; n_{x_{r_x}})$, for $x = 0$ or 1, is a partition of n (by positive integers).

If $\overline{n}_0 = \overline{n}_1 = n$, then $\mathscr{S}(\bar{n}_0, \bar{n}_1) = \mathrm{M}_n(\mathrm{C}[0,1])$.

In general, a splitting interval algebra $A = \mathscr{S}(\bar{n}_0, \bar{n}_1)$ is a continuous field of C*-algebras over $[0,1]$, whose fibre $A(x)$ at x is the full matrix algebra $\mathrm{M}_n(\mathbb{C})$ unless $x = 0$ or 1, where the fibres are

$$A(0) = \bigoplus_{i=1}^{r_0} \mathrm{M}_{n_{0_i}}(\mathbb{C}), \quad \text{and} \quad A(1) = \bigoplus_{i=1}^{r_1} \mathrm{M}_{n_{1_i}}(\mathbb{C}),$$

respectively. $x = 0$ or 1 is called a broken endpoint if $r_x > 1$, i.e., if $A(x)$ is not $\mathrm{M}_n(\mathbb{C})$.

It is easy to check that the spectrum of A, $\mathrm{SP}(A)$ is $\{0_1,\cdots,0_{r_0}\}\cup(0,1)\cup\{1_1,\cdots,1_{r_1}\}$. If $x=0$ or 1 is a broken endpoint, then any of $\{x_i : 1\le i\le r_x\}$ is called a fractional endpoint (or fractional point for short).

In the inductive system $(A_n,\phi_{n,m})$, where $\phi_{n,m}=\phi_{m-1,m}\circ\phi_{m-2,m-1}\circ\cdots\circ\phi_{n,n+1}$ are all homomorphisms. If $A_n=\bigoplus\limits_i A_n^i$, we shall use $\phi_{n,m}^{i,j}$ to denote the partial map of $\phi_{n,m}$ from the i-th block A_n^i of A_n to the j-th block A_m^j of A_m. Necessarily, we can assume $\phi_{n,n+1}(1_{A_n^i})\neq 0$; otherwise, we can simply delete A_n^i from A_n without causing any change to the limit algebra.

Definition 1.1.2. *For a unital C^*-algebra A, let TA denote the set of tracial states of A, i.e. τ is in TA if and only if τ is a positive linear map from A to the complex plane $\mathbb{C}$ with $\tau(xy)=\tau(yx)$ and $\tau(id)=1$. AffTA is the collection of all affine maps from TA to $\mathbb{C}$ (In most references, AffTA is defined to be the set of all affine maps from TA to $\mathbb{R}$. Our AffTA is a complexification of the standard AffTA).*
An element $1\in$ AffTA defined by $1(\tau)=1$ for all $\tau\in TA$ will be called the unit of AffTA. AffTA together with the positive cone $AffTA_+$ and the unit element 1 form a scaled ordered complex Banach space.
(Note that for any element $x\in$ AffTA, there are x_1,x_2,x_3,x_4 in $AffTA_+$ such that $x=x_1-x_2+i(x_3-x_4)$.)

Definition 1.1.3. *For a unital C^*-algebra A, let $\bigvee(A)$ denote the collection of all Murray-von Neumann equivalence classes of projections in $M_\infty(A)(=\bigcup\limits_{n=1}^{\infty}M_n(A))$. Define*

$$K_0(A)=\{(a,b): a\in\bigvee(A),\ b\in\bigvee(A)\}/\sim,$$

where $(a,b)\sim(a',b')$ if and only if there exists $c\in\bigvee(A)$ such that

$$a+b'+c=a'+b+c\in\bigvee(A).$$

For each C^*-algebra A, define the scale of A to be the subset

$$\sum A\triangleq\{[p]\in K_0(A): p \text{ is a projection of } A\},$$

and

$$K_0(A)_+\triangleq\{[p]\in K_0(A): p \text{ is a projection of } M_\infty(A)\}.$$

If A is stably finite, then

$$\mathrm{K}_0(A)_+-\mathrm{K}_0(A)_+=\mathrm{K}_0(A),\quad \mathrm{K}_0(A)_+\cap(-\mathrm{K}_0(A)_+)=0.$$

Every homomorphism $\Lambda: A\to B$ induces a homomorphism between scaled ordered groups,

$$(\mathrm{K}_0(A),\mathrm{K}_0(A)_+,\sum A)\longrightarrow(\mathrm{K}_0(B),\mathrm{K}_0(B)_+,\sum B)$$

in the sense that $\mathrm{K}_0(\Lambda)(\mathrm{K}_0(A)_+)\subset\mathrm{K}_0(B)_+$ and $\mathrm{K}_0(\Lambda)(\sum A)\subset\sum B$.

Definition 1.1.4. *The map* $< \cdot\ ,\ \cdot >: TA \times K_0(A) \to \mathbb{R}$ *is defined by*

$$< \tau, x >= \sum_{i=1}^{k} \tau(p_{ii}) - \sum_{i=1}^{k} \tau(q_{ii}),$$

where $\tau \in TA,\ x = [p] - [q] \in K_0(A)$ *is represented by the formal difference of two projections* $p, q \in M_k(A)$*. Set*

$$x(\tau) \triangleq \tau(x),\ \textit{for } x \in K_0(A).$$

In this way, each element x *in* $K_0(A)$ *induces an affine map from* TA *to* $\mathbb{R}$*, and therefore, defines an element of* $AffTA$*. This determines a map* $\sigma : K_0(A) \to AffTA$ *by* $\sigma(x)(\tau) = \tau(x)$*.*

Let $\alpha : K_0(A) \to K_0(B)$ *be a scaled ordered group homomorphism, and let* $\xi : TB \to TA$ *be an affine map. Then* ξ *induces a linear map*

$$\xi^* : AffTA \to AffTB$$

defined by

$$\xi^*(f)(\tau) = f(\xi(\tau))$$

for all $f \in AffTA$ *and* $\tau \in TB$*. It is obvious that*

$$\xi^*(AffTA_+) \subset AffTB_+,\ \xi^*(1) = 1.$$

Hence, ξ *induces a positive unital linear map (or scaled ordered map) from* $AffTA$ *to* $AffTB$*.*

We shall say that α *and* ξ *are compatible if*

$$\tau(\alpha(x)) = (\xi(\tau))(x), \quad \textit{for } \forall x \in K_0(A), \quad \tau \in TB.$$

It is evident that α *and* ξ *are compatible if and only if the following diagram commutes*

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\sigma} & AffTA \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\xi^*} \\ K_0(B) & \xrightarrow{\sigma} & AffTB \end{array}$$

In the rest of this paper, we will use ξ *to denote* ξ^**.*

Any unital homomorphism $\phi : A \to B$ induces a unital positive linear map

$$\mathrm{AffT}\phi : \mathrm{AffT}A \to \mathrm{AffT}B.$$

Suppose that $P \in \mathrm{M}_l(\mathrm{C}(X))$ is a non-zero projection with constant rank. It is well known that

$$\mathrm{AffT}(PM_l(\mathrm{C}(X))P) = \mathrm{AffT}(\mathrm{M}_l(\mathrm{C}(X))) = \mathrm{C}(X).$$

Lemma 1.1.5. *(1) If $\phi : C(X) \to M_l(C(Y))$ is a unital homomorphism, then $\mathit{Aff}T\phi : C(X) \to C(Y)$ is given by*

$$\mathit{Aff}T\phi(f) = \frac{1}{l}\sum_{i=1}^{l}\phi(f)_{ii},\ \forall f \in C(X),$$

where $\phi(f)_{ii}$ denotes the entry of $\phi(f) \in M_l(C(Y))$ at the position (i,i).

(2) Let $\phi_1 : C(X) \to PM_{l_1}(C(Y))P,\quad \phi_2 : C(X) \to QM_{l_2}(C(Y))Q$ be two unital homomorphisms, and

$$\phi = \mathrm{diag}(\phi_1, \phi_2) : C(X) \to (P \oplus Q)M_{l_1+l_2}(C(Y))(P \oplus Q).$$

Then

$$\mathit{Aff}T\phi = \frac{k_1}{k_1+k_2}\mathit{Aff}T\phi_1 + \frac{k_2}{k_1+k_2}\mathit{Aff}T\phi_2,$$

where $k_1 = \mathrm{rank}(P)$ and $k_2 = \mathrm{rank}(Q)$. Also, if P and Q are two orthogonal projections in $M_l(C(Y))$, then $\phi = \mathrm{diag}(\phi_1,\phi_2)$ can be considered to be a homomorphism from $C(X)$ to $(P+Q)M_l(C(Y))(P+Q)$ and the above equality still holds.

Let $\psi : \mathrm{C}(X) \to PM_{k_1}(\mathrm{C}(Y))P$ be a unital homomorphism. For any fixed point $y \in Y$, there are points $x_1(y), x_2(y), \cdots, x_k(y) \in X$, and a unitary $U_y \in \mathrm{M}_{k_1}(\mathrm{C}(Y))$ such that

$$\psi(f)(y) = P(y)U_y \begin{pmatrix} f(x_1(y)) & & & & & \\ & \ddots & & & & \\ & & f(x_k(y)) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} U_y^* P(y)$$

for all $f \in \mathrm{C}(X)$. Equivalently, there are k rank one orthogonal projections $P_1, P_2, \cdots, P_k$ with $\sum_{i=1}^{k} P_i(y) = P(y)$ and $x_1(y), \cdots, x_k(y) \in X$, such that

$$\psi(f)(y) = \sum_{i=1}^{k} f(x_i(y))P_i(y), \forall f \in \mathrm{C}(X).$$

Definition 1.1.6. *Let $\mathrm{SP}\psi_y$ denote the set $\{x_1(y), x_2(y), \cdots, x_k(y)\}$, counting multiplicities. In other words, if a point is repeated in the diagonal of the above matrix, it is included with the same multiplicity in $\mathrm{SP}\psi_y$. We call $\mathrm{SP}\psi_y$ the spectrum of ψ at the point y (see also [43]), and define the spectrum of ψ, denoted by $\mathrm{SP}\psi$, to be the closed subset*

$$\mathrm{SP}\psi \triangleq \overline{\bigcup_{y\in Y}\mathrm{SP}\psi_y} \subseteq X.$$

Alternatively, $\mathrm{SP}\psi$ is the complement of the spectrum of the kernel of ψ, considered as a closed ideal of $\mathrm{C}(X)$. The map ψ can be factored as

$$\mathrm{C}(X) \xrightarrow{i^*} \mathrm{C}(\mathrm{SP}\psi) \xrightarrow{\psi_1} PM_{k_1}(\mathrm{C}(Y))P$$

with ψ_1 an injective homomorphism, where i denotes the inclusion map $\mathrm{SP}\psi \hookrightarrow X$.

Also, if $A = PM_{k_1}(\mathrm{C}(Y))P$, then we call the space Y the spectrum of the algebra A, and denote $\mathrm{SP}A = Y (= \mathrm{SP}(id))$.

If we group all the repeated points in $\{x_1(y), x_2(y), \cdots, x_k(y)\}$, and sum their corresponding projections, we can obtain

$$\psi(f)(y) = \sum_{i=1}^{l} f(\lambda_i(y))P_i, \quad (l \leq k),$$

where $\{\lambda_1(y), \lambda_2(y), \cdots, \lambda_l(y)\}$ is equal to $\{x_1(y), x_2(y), \cdots, x_k(y)\}$ as a set, but $\lambda_i(y) \neq \lambda_j(y)$ when $i \neq j$; and each P_i is the sum of the projections corresponding to $\lambda_i(y)$. If $\lambda_i(y)$ has multiplicity m (i.e. it appears m times in $\{x_1(y), x_2(y), \cdots, x_k(y)\}$), then $\mathrm{rank}(P_i) = m$.

We call the projection P_i the spectral projection of ψ at y with respect to the spectral element $\lambda_i(y)$. If $X_1 \subset X$ is a subset of X,

$$\sum_{x_i(y) \in X_1} P_i$$

is called the spectral projection of ψ at y corresponding to the subset X_1.

Let $\phi : \mathrm{M}_k(\mathrm{C}(X)) \to PM_l(\mathrm{C}(Y))P$ be a unital homomorphism. Set $\phi(e_{11}) = p$, where e_{11} is the canonical matrix unit corresponding to the upper left corner. Set

$$\phi_1 = \phi|_{e_{11}\mathrm{M}_k(\mathrm{C}(X))e_{11}} : \mathrm{C}(X) \to pM_l(\mathrm{C}(Y))p.$$

Then $PM_l(\mathrm{C}(Y))P$ can be identified with $pM_l(\mathrm{C}(Y))p \otimes \mathrm{M}_k$ in such a way that

$$\phi = \phi_1 \otimes \mathrm{id}_k.$$

Define

$$\mathrm{SP}\phi_y := \mathrm{SP}(\phi_1)_y, \quad \mathrm{SP}\phi := \mathrm{SP}\phi_1.$$

It is well-known that for homomorphisms ϕ and ϕ_1 as defined above with $\mathrm{rank}p = k$,

$$\mathrm{AffT}\phi_1(f)(y) = \frac{1}{k} \sum_{x_i(y) \in \mathrm{SP}(\phi_1)_y} f(x_i(y))$$

and that

$$\mathrm{AffT}\phi = \mathrm{AffT}\phi_1.$$

Let $\phi : \mathrm{M}_k(\mathrm{C}(X)) \to PM_l(\mathrm{C}(Y))P$ be a (not necessary unital) homomorphism, where X and Y are connected finite simplicial complexes. Then

$$\sharp(\mathrm{SP}\phi_y) = \frac{\mathrm{rank}\phi(1_k)}{\mathrm{rank}(1_k)}, \quad \forall y \in Y,$$

where $\sharp(\cdot)$ denotes the number of elements in the set, counting multiplicities. It is also true that for any non-zero projection $p \in \mathrm{M}_k(\mathrm{C}(X))$,

$$\sharp(\mathrm{SP}\phi_y) = \frac{\mathrm{rank}\phi(p)}{\mathrm{rank}(p)}.$$

Let

$$\phi : A = \bigoplus_{i=1}^{q} \mathrm{M}_{k_i}(\mathrm{C}(X_i)) \to B = \bigoplus_{j=1}^{t} P_j M_{l_j}(\mathrm{C}(Y_j))P_j$$

be a homomorphism and denote by Y the disjoint union $\coprod Y_j$ of the spaces $\{Y_j\}_{j=1}^t$. For each $y \in Y$, $y \in Y_j$ for some j, the spectrum of the homomorphism ϕ at the point $y \in Y$ is defined by $\mathrm{SP}\phi_y = \bigcup_{i=1}^{q} \mathrm{SP}(\phi^{i,j})_y$, where $\phi^{i,j} : A^i := \mathrm{M}_{k_i}(\mathrm{C}(X_i)) \to \phi(1_{A_i})P_j M_{l_j}(\mathrm{C}(Y_j))P_j\phi(1_{A_i})$ is the partial map of ϕ. Note that

$$\mathrm{SP}\phi_y = \bigcup_{i=1}^{q} \mathrm{SP}(\phi^{i,j})_y \subset X := \coprod X_i.$$

For any $f \in \mathrm{AffT}A^i = \mathrm{C}(X_i)$,

$$\mathrm{AffT}\phi^{i,j}(f) = \frac{\mathrm{rank}P_j}{\mathrm{rank}(\phi^{i,j}(\mathbf{1}_{A_i}))}(\mathrm{AffT}\phi(f))_j,$$

where the AffT map on the left hand side is taken by regarding the homomorphism $\phi^{i,j}$ as a map from A^i to $\phi^{i,j}(\mathbf{1}_{A^i})B\phi^{i,j}(\mathbf{1}_{A^i})$, and the AffT map on the right hand side is taken by regarding the homomorphism ϕ as a map from A to B^j, the j-th summand of B, where $B^j = P_j M_{l_j}(\mathrm{C}(Y_j))P_j$.

Definition 1.1.7. *For any $\eta > 0$, $\delta > 0$, a unital homomorphism $\phi : C(X) \to QM_k(C(Y))Q$ is said to have the property sdp(η, δ) (spectral distribution property with respect to η and δ), if for any η-open ball $B_\eta(x) \triangleq \{x' \in X : dist(x', x) < \eta\} \subset X$ and any point $y \in Y$, $\sharp(\mathrm{SP}\phi_y \cap B_\eta(x)) \geq \delta \cdot \sharp(\mathrm{SP}\phi_y)$, counting multiplicities.*

For a unital homomorphism $\phi : PM_k(C(X))P \to QM_l(C(Y))Q$, we say that ϕ has the property sdp$(\cdot,\cdot)$, if

$$\phi|_{pM_k(C(X))p} : C(X)(\cong pM_l(C(X))p) \to \phi(p)M_l(C(Y))\phi(p)$$

has the property sdp$(\cdot,\cdot)$, where P and Q are non-zero projections and p is a rank 1 sub-projection of P.

For a unital homomorphism $\phi : A \to B$, where $A \subseteq M_n(C(X)), B \subseteq M_m(C(Y))$ are splitting interval algebras, $X = Y = [0,1]$, ϕ has the property sdp(η, δ), if

$$\sharp(\mathrm{SP}\phi_y \cap B_\eta(x)) \geq \delta \cdot \sharp(\mathrm{SP}\phi_y),$$

(counting multiplicity), for any $B_\eta(x) \subseteq X$, any $y \in Y$.

Lemma 1.1.8. *Let A be a unital C^*-algebra, $q \in A$ be a non-zero projection. If $k[q] = l[1_A]$ in $K_0(A)$, then*

$$\mathit{Aff}Ti(f) = \frac{l}{k} f, \quad \forall f \in \mathit{Aff}TqAq,$$

where 1_A is the unit of A and $i : qAq \to A$ is the embedding map. In particular, for the interval algebra $A = M_n(C(X))$, $X = [0,1]$, let $q \in A$ be a non-zero projection. We then have

$$\mathit{Aff}Ti(g) = \frac{\mathrm{rank}q}{n} g, \quad \forall g \in \mathit{Aff}T(qM_n(C(X))q).$$

Lemma 1.1.9. *(see Lemma 1.2 of [31]) Let A be a splitting interval algebra. Let $Q_{x_i} : A \to A(x_i)$ be the canonical evaluation map at an endpoint x_i for $x = 0$ or 1.*

(1) The direct sum $\bigoplus_{x_i} (Q_{x_i})_* : K_0(A) \to \mathbb{Z}^{r_0+r_1}$ is an injective morphism, and identifies $K_0(A)$ with the subgroup of $\mathbb{Z}^{r_0+r_1}$:

$$\{(\bar{k}_0, \bar{k}_1) \in \mathbb{Z}^{r_0} \times \mathbb{Z}^{r_1} : \sum_{i=1}^{r_0} k_{0_i} = \sum_{i=1}^{r_1} k_{1_i}\},$$

with the inherited order from the standard order of $\mathbb{Z}^{r_0+r_1}$, where the k_{x_i}'s are coordinates of $\bar{k}_x$. Also, $[1] \cong (\bar{n}_0; \bar{n}_1)$;

(2) $\mathrm{K}_1(A) = \{0\}$.

In particular, $K_0(\varphi(\bar{n}_0; \bar{n}_1)) \cong \mathbb{Z}^{r_0+r_1-1}$ is a group.

Lemma 1.1.10. *(see Lemma 1.3 of [31])*

(1) Any Radon probability measure μ on [0,1] defines a tracial state on A in the following way

$$\mu(f) = \int \mathrm{Tr}(f) d\mu, \quad \text{for } f \in A,$$

where Tr is the normalized canonical trace on $M_n(\mathbb{C})$, the generic fibre of A. The corresponding tracial state will be denoted again by μ.

(2) Any x_i defines a point-mass tracial state on A,

$$\delta_{x_i}(f) = \mathrm{Tr}_{A(x_i)}(Q_{x_i}(f)),$$

where $\mathrm{Tr}_{A(x_i)}$ is the normalized trace on $A(x_i)$, $x = 0$ or 1.

(3) $\delta_0 = \sum_i \frac{n_{0_i}}{n} \delta_{0_i}$, $\delta_1 = \sum_i \frac{n_{1_i}}{n} \delta_{1_i}$, where δ_{0_i}, δ_{1_i} are defined in (2).

Lemma 1.1.11. *(see Lemma 1.4 of [31])*

(1) Any $t \in \mathrm{T}(A)$ *determines uniquely a vector* $\lambda(t) = (\bar{\lambda}_0; \bar{\lambda}_1; \lambda) \in \mathbb{R}_+^{r_0} \times \mathbb{R}_+^{r_1} \times \mathbb{R}_+$ *with* $\bar{\lambda}_x = (\lambda_{x_1}, \cdots, \lambda_{x_{r_x}})$*, where* $x = 0$ *or* 1 *such that*

$$\min\{\lambda_{0_i} : 1 \leq i \leq r_0\} = \min\{\lambda_{1_i} : 1 \leq i \leq r_1\} = 0,$$

and

$$t = \sum_{x \in \{0,1\}} \sum_{i=1}^{r_x} \lambda_{x_i} \cdot \delta_{x_i} + \lambda \cdot \mu,$$

where μ *is a Radon probability measure on [0,1] (see Lemma 1.1.10), which is determined by* t *if* $\lambda \neq 0$*. We shall call this equality the standard form of* t*, and we shall also call* μ *the principal part of* t *and the rest, the residual part.*

(2) Any $f \in \mathrm{AffT}(A)$ *defines a real-valued function* $\delta^*(f)$ *on* $\mathrm{SP}(A)$*:*

$$\delta^*(f)(x) = f(\delta_x).$$

The map δ^* identifies $\mathrm{AffT}A$, as an ordered space with unit, with the space of all real-valued functions f on $\mathrm{SP}(A)$ satisfying the following
(i) f is continuous on (0,1),
(ii) $x = 0$ *or* 1, $\lim\limits_{[\bar{x}] \to x} f([\bar{x}]) = \sum\limits_i \frac{n_{x_i}}{n} f(x_i)$, where the limit is taken in [0,1] (Recall that $[\cdot] : \mathrm{SP}(A) \to [0,1]$ is the canonical quotient map).

Theorem 1.1.12. *(see [60] Theorem 3.1). Let A, B be two splitting interval algebras. For any unital homomorphism* $\phi : A \to B$*, any finite set* $F \subset A$ *and any* $\varepsilon > 0$*, there exists a unital map* $\phi' : A \to B$ *such that*
(1) $||\phi(f) - \phi'(f)|| < \varepsilon, \ \forall f \in F$,
(2) There exist continuous maps $\{s_j(\cdot)\}_{j=1}^p \subset C([0,1]; \mathrm{SP}(A))$ *and a unitary* $W \in M_m(C[0,1])$*, where* M_m *is the generic fibre of* B*, such that*

$$\phi'(f)(y) = W(y) diag(f(s_1(y)), \cdots, f(s_p(y))) W^*(y) \qquad (1.1)$$

for all $f \in A, y \in [0,1]$.

Any homomorphism of the form (1.1) is called standard and the $\{s(\cdot)\}_{j=1}^p$ are its eigenvalue maps. More generally, a homomorphism between two finite direct sums of splitting interval algebras is called standard, if on each summand of B the map assumes the form (1.1), where the $\{s(\cdot)\}_{j=1}^p$ are now continuous maps form $[0,1]$ to the spectrum of A. Furthermore, by Theorem 4 of [4], we can assume that for each $y \in [0,1]$, the eigenvalues $s_i(y)$ are distinct on $(0,1)$–that is if $s_i(y), \ s_j(y) \in (0,1)$ and $i \neq j$, then $s_i(y) \neq s_j(y)$ (see page 67 of [31]).

Lemma 1.1.13. *Let A be a splitting interval algebra, and $q \in A$ be a non-zero projection. Then based on 1.1.7 and 1.1.10, we have*

$$AffTI(f)(0_i) = \frac{k_{0_i}}{n_{0_i}} f(0_i), \quad AffTI(f)(1_j) = \frac{k_{1_j}}{n_{1_j}} f(1_j),$$

where $k_{0_i} \neq 0, k_{1_j} \neq 0$, and

$$AffTI(f)(x) = \frac{rankq}{n} f(x), \qquad \text{if } x \in [0,1] \text{ is not fractional,}$$

where $k_0(q) = (\overline{k}_0, \overline{k}_1) = (k_{0_1}, k_{0_2}, \cdots, k_{0_{r_0}}, k_{1_1}, k_{1_2}, \cdots, k_{1_{r_1}})$ and $I : qAq \to A$ is the embedding map.

1.2 Existence Theorem

In this section, we establish the Existence Theorem which will be used in the proof of the main theorem of this chapter. Our approach is based on the analysis of the properties of homomorphisms between two splitting interval algebras.

Lemma 1.2.1. *Let $A = \lim\limits_{n\to\infty}(A_n, \phi_{n,m})$, $B = \lim\limits_{n\to\infty}(B_n, \psi_{n,m})$ be C^*-algebras where A_n, B_n are splitting interval algebras, and let $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ be a scaled ordered group isomorphism. Then there exist subsequences $A_{n_1}, A_{n_2}, \cdots, A_{n_i}, \cdots$ and $B_{m_1}, B_{m_2}, \cdots, B_{m_i}, \cdots$ and scaled ordered K_0 maps $\alpha_i : \mathrm{K}_0(A_{n_i}) \to \mathrm{K}_0(B_{m_i})$ and $\beta_i : \mathrm{K}_0(B_{m_i}) \to \mathrm{K}_0(A_{n_{i+1}})$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc}
\mathrm{K}_0A_{n_1} & \xrightarrow{\mathrm{K}_0\phi_{n_1,n_2}} & \mathrm{K}_0A_{n_2} & \xrightarrow{\mathrm{K}_0\phi_{n_1,n_2}} & \mathrm{K}_0A_{n_3} & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\
\downarrow{\scriptstyle \alpha_1} & \nearrow{\scriptstyle \beta_1} & \downarrow{\scriptstyle \alpha_2} & \nearrow{\scriptstyle \beta_2} & \downarrow{\scriptstyle \alpha_3} & \nearrow{\scriptstyle \beta_3} & & & \downarrow{\scriptstyle \alpha} \\
\mathrm{K}_0B_{m_1} & \xrightarrow[\mathrm{K}_0\psi_{m_1,m_2}]{} & \mathrm{K}_0B_{m_2} & \xrightarrow[\mathrm{K}_0\psi_{m_2,m_3}]{} & \mathrm{K}_0B_{m_3} & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B.
\end{array}$$

That is

$$\beta_i \circ \alpha_i = \mathrm{K}_0\phi_{n_i,n_{i+1}}, \quad \alpha_{i+1} \circ \beta_i = \mathrm{K}_0\psi_{m_i,m_{i+1}},$$

$$\alpha \circ \mathrm{K}_0\phi_{n_i,\infty} = \mathrm{K}_0\psi_{m_i,\infty} \circ \alpha_i, \quad \alpha^{-1} \circ \mathrm{K}_0\psi_{m_i,\infty} = \mathrm{K}_0\phi_{n_{i+1},\infty} \circ \beta_i.$$

For convenience, from now on, we assume that $n_i = i$ and $m_i = i$. For scaled ordered K_0 maps $\alpha_i : \mathrm{K}_0A_i \to \mathrm{K}_0B_i$, $\beta_i : \mathrm{K}_0B_i \to \mathrm{K}_0A_{i+1}$, there exist homomorphisms $\widetilde{\Lambda}_i : A_i \to B_i$, $\widetilde{\mathcal{M}}_i : B_i \to A_{i+1}$ such that $\mathrm{K}_0(\widetilde{\Lambda}_i) = \alpha_i$, $\mathrm{K}_0(\widetilde{\mathcal{M}}_i) = \beta_i$.
Lemma 1.2.1 comes from the work of steps 1 and 2 of [[60], Theorem 8.3].

Lemma 1.2.2. *Let $A = \lim\limits_{n\to\infty}(A_n, \phi_{n,m})$, $B = \lim\limits_{n\to\infty}(B_n, \psi_{n,m})$, where A_n, B_n are splitting interval algebras. Let $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ be a scaled ordered group isomorphism, and $\xi : \mathrm{AffT}A \to \mathrm{AffT}B$ be an isomorphism of scaled ordered complete Banach spaces* *which is compatible with α. For any A_n, any given finite set $F \subseteq \mathrm{AffT}A_n$, and any $\varepsilon > 0$, there exist $m > n$ and a map $\xi_n : \mathrm{AffT}A_n \to \mathrm{AffT}B_m$ such that*

$$||\mathrm{AffT}\psi_{m,\infty} \circ \xi_n(f) - \xi \circ \mathrm{AffT}\phi_{n,\infty}(f)|| < \varepsilon, \ \forall f \in F.$$

In particular, ξ_n can be chosen to be compatible with $\mathrm{K}_0\psi_{n,m} \circ \alpha_n$, where α_n is as described in Lemma 1.2.1.

Proof. The idea of this proof is inspired by the work of Lemma 6.6 in [34] , but our case is much more complicated.

Without loss of generality, we may assume that $n = 1$ and A_1 has only two blocks, i.e. $A_1 = \bigoplus_{k=1}^{2} A_1^k \subseteq \bigoplus_{k=1}^{2} \mathrm{M}_{[1,k]}(\mathrm{C}(X_k))$, where $X_k = [0,1], k = 1,2$.

$$\mathrm{SP}(A_1^k) = (0,1) \cup \{0_1^k, \cdots, 0_{r_0(k)}^k\} \cup \{1_1^k, \cdots, 1_{r_1(k)}^k\},$$

where $r_0(k)$ and $r_1(k)$ are as defined in 1.1, and $k = 1,2$ stands for different splitting algebras A_1^k. For $\varepsilon_1 > 0$ (to be determined) and any finite set $F \subseteq \mathrm{AffT}A_1$, there exists $\delta_1 > 0$ ($\delta_1 < \frac{1}{4}$), such that $|f(x_1) - f(x_2)| < \varepsilon_1$, for all $f \in F$, $x_1, x_2 \in X_k$ and $|x_1 - x_2| < \delta_1$. (2.1)

For X_k, let $U_{k_1}, U_{k_2}, \cdots, U_{k_{\Lambda_k}}$ be a finite open cover of X_k such that $U_{k_t} = \{x \in X_k : dist(x, x_{k_t}) < \delta_1\}$ and $x_{k_1} = 0$, $x_{k_{\Lambda_k}} = 1$, for certain points $x_{k_t} \in X_k$, $k = 1,2$, $t = 1,2,\cdots,\Lambda_k$. Let $\{h_{k_t}\}$ be the partition of unity subordinate to U_{k_t}. Then $\sum_{t=1}^{\Lambda_k} h_{k_t} = \chi_{X_k}$, where χ_{X_k} denotes the characteristic function corresponding to the set $X_k \subset X = X_1 \sqcup X_2$, $k = 1,2$.

Let $\pi : \mathrm{AffT}A_1 \to \mathbb{C}^\Lambda$ be defined as

$$\pi(f) = \sum_{k,t} f(x_{k_t}) v_{k_t} + \sum_{k=1}^{2} \sum_{x \in \{0,1\}} \sum_{i=1}^{r_x(k)-1} f(x_i^k) u_{i,x}^k,$$

where $\Lambda = \Lambda_1 + \Lambda_2 + r_0(1) - 1 + r_0(2) - 1 + r_1(1) - 1 + r_1(2) - 1$, and $\{v_{k_t}, u_{i,x}^k\}$ is the standard basis of $\mathbb{C}^\Lambda$. Finally let $\tilde{\pi} : \mathbb{C}^\Lambda \to \mathrm{AffT}A_1$ be defined by

$$\tilde{\pi}(\sum_{k,t} a_{k_t} v_{k_t} + \sum_{k=1}^{2} \sum_{x \in \{0,1\}} \sum_{i=1}^{r_{x(k)}-1} b_{i,x}^k u_{i,x}^k)(y)$$

$$= \begin{cases} \sum_t a_{k_t} h_{k_t}(y), & y \in X_k \text{ not fractional}, \ k = 1,2 \\ b_{i,0}^k, & y = 0_i^k, \ i = 1,\cdots,r_0(k)-1, \ k = 1,2 \\ b_{i,1}^k, & y = 1_i^k, \ i = 1,\cdots,r_1(k)-1, \ k = 1,2 \\ \{\sum_t a_{k_t} h_{k_t}(0) - \sum_{i=1}^{r_0(k)-1} \frac{n_{0_i^k}}{n(k)} \cdot b_{i,0}^k\} \cdot \frac{n(k)}{n_{0_{r_0(k)}^k}}, & y = 0_{r_0(k)}^k, \ k = 1,2 \\ \{\sum_t a_{k_t} h_{k_t}(1) - \sum_{i=1}^{r_1(k)-1} \frac{n_{1_i^k}}{n(k)} \cdot b_{i,1}^k\} \cdot \frac{n(k)}{n_{1_{r_1(k)}^k}}, & y = 1_{r_1(k)}^k, \ k = 1,2. \end{cases}$$

Here $n_{x_i^k}$ stands for the size of the block of A_k at the fractional point x_i^k, $x=0,1,\ k=1,2,\ i=1,\cdots,r_x(k)$, and $n(k)=[1,k]$ is the size of A_k.

By (2.1), for all $f\in F$, $||\tilde{\pi}\circ\pi(f)-f||<\varepsilon_1$.

Let $E_k,\ F_k^i,\ G_k^i\in \mathrm{AffT}A_1$ be defined by

$$E_k(x)=\begin{cases}1, & x\in \mathrm{SP}(A_1^k)\\ 0, & \text{other}\end{cases},F_k^i(x)\begin{cases}1, & x=0_i^k\\ -\frac{n_{0_i^k}}{n_{0_{r_0}^k}}, & x=0_{r_0}^k\\ 0, & \text{other}\end{cases},G_k^j(x)\begin{cases}1, & x=1_j^k\\ -\frac{n_{1_j^k}}{n_{1_{r_1}^k}}, & x=1_{r_1}^k\\ 0, & \text{other}\end{cases},$$

where $k=1,2,\ i=1,\cdots,r_0(k)-1,\ j=1,\cdots,r_1(k)-1$. Then we have

$$\pi(E_k)=\sum_t v_{k_t}+\sum_{x\in\{0,1\}}\sum_{i=1}^{r_x(k)-1}u_{i,x}^k,\quad \tilde{\pi}(\pi(E_k))=E_k,$$

$$\tilde{\pi}(\pi(F_k^i))=F_k^i\ \text{ and }\ \tilde{\pi}(\pi(G_k^i))=G_k^i.$$

Notice that

$$\mathscr{R}\triangleq\{\pi(E_1),v_{1_2},\cdots,v_{1_{\Lambda_1}},\pi(E_2),v_{2_1},\cdots,v_{2_{\Lambda_2}}\}\cup\{\bigcup_{k,i}\pi(F_k^i)\}\cup\{\bigcup_{k,i}\pi(G_k^i)\}$$

is another basis of $\mathbb{C}^\Lambda$.

For $\varepsilon_1>0$ and the finite set $\pi(F)\subset\mathbb{C}^\Lambda$, there exists $\delta_2>0$ such that for any two linear maps $L_i:\mathbb{C}^\Lambda\to W$ ($i=1,2$, W is an arbitrary normed linear space), the condition $||L_1(x)-L_2(x)||<\delta_2$, for all $x\in\mathscr{R}$ implies that

$$||L_1(f)-L_2(f)||<\varepsilon_1,\quad \text{for all } f\in\pi(F).\qquad (2.2)$$

If we choose $m>1$ large enough, then for each $\tilde{\pi}(v_{k_t})=h_{k_t}\in\mathrm{AffT}A_1$, there exists $g_{k_t}\in\mathrm{AffT}B_m$ such that

$$||\xi\circ\mathrm{AffT}\phi_{1,\infty}(h_{k_t})-\mathrm{AffT}\psi_{m,\infty}(g_{k_t})||<\delta_2,$$

where $1\le k\le 2$, $2\le t\le\Lambda_k$. Define $\bar{\xi}(v_{k_t})=g_{k_t}$, $2\le t\le\Lambda_k$, $1\le k\le 2$.

For $\pi(E_k)$, $\pi(F_k^i)$, $\pi(G_k^i)$, we know that

$$\tilde{\pi}(\pi(E_k))=E_k\in\mathrm{K}_0A_1\subset\mathrm{AffT}A_1,$$

$$\tilde{\pi}(\pi(F_k^i))=F_k^i\in\mathrm{K}_0A_1\subset\mathrm{AffT}A_1,$$

$$\tilde{\pi}(\pi(G_k^i))=G_k^i\in\mathrm{K}_0A_1\subset\mathrm{AffT}A_1,$$

and $\alpha'(E_k)\in\mathrm{K}_0B_m\subset\mathrm{AffT}B_m$, where $\alpha'=\mathrm{K}_0\psi_{1,m}\circ\alpha_1$.

Consequently, we define

$$\bar{\xi}_1(\pi(E_k))=\alpha'(E_k),\quad k=1,2;$$

$$\bar{\xi}_1(\pi(F_k^i)) = \alpha'(F_k^i), \quad i = 1, 2, \cdots, r_0(k) - 1, \quad k = 1, 2;$$
$$\bar{\xi}_1(\pi(G_k^i)) = \alpha'(G_k^i), \quad i = 1, 2, \cdots, r_1(k) - 1, \quad k = 1, 2.$$

Then

$$\begin{aligned} & ||\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi}(v_{k_t}) - \mathrm{AffT}\psi_{m,\infty} \circ \bar{\xi}_1(v_{k_t})|| \\ = & ||\xi \circ \mathrm{AffT}\phi_{1,\infty}(h_{k_t}) - \mathrm{AffT}\psi_{m,\infty}(g_{k_t})|| < \delta_2, \end{aligned}$$

where $2 \leq t \leq \Lambda_k, \quad 1 \leq k \leq 2$.

Let $G = \{E_k : \ k = 1, 2\} \bigcup \{F_k^i : \ i = 1, 2, \cdots, r_0(k) - 1, \ k = 1, 2\} \bigcup \{G_k^i : \ i = 1, 2, \cdots, r_1(k) - 1, \ k = 1, 2\}$. Then for $\forall E \in G$, we have

$$\begin{aligned} & ||\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi}(\pi(E)) - \mathrm{AffT}\psi_{m,\infty} \circ \bar{\xi}_1(\pi(E))|| \\ = & ||\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi}(\pi(E)) - \mathrm{AffT}\psi_{m,\infty} \circ \alpha'(E)|| \\ = & ||\xi \circ \mathrm{AffT}\phi_{1,\infty}(E) - \mathrm{K}_0\psi_{m,\infty}(\alpha'(E))|| \\ = & ||\alpha \circ \mathrm{K}_0\phi_{1,\infty}(E) - \mathrm{K}_0\psi_{m,\infty}(\alpha'(E))|| = 0 < \delta_2. \end{aligned}$$

We now extend the definition of $\bar{\xi}_1$ from $\mathscr{R}$ to the whole of $\mathbb{C}^\Lambda$ linearly, then by (2.2), we have

$$||\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi}(f) - \mathrm{AffT}\psi_{m,\infty} \circ \bar{\xi}_1(f)|| < \varepsilon_1, \quad \forall f \in \pi(F).$$

Let $\xi_1 = \bar{\xi}_1 \circ \pi$, then for each $f \in F$,

$$\begin{aligned} & ||(\mathrm{AffT}\psi_{m,\infty} \circ \xi_1)(f) - (\xi \circ \mathrm{AffT}\phi_{1,\infty})(f)|| \\ \leq & ||(\mathrm{AffT}\psi_{m,\infty} \circ \xi_1)(f) - (\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi} \circ \pi)(f)|| \\ & + ||(\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi} \circ \pi)(f) - (\xi \circ \mathrm{AffT}\phi_{1,\infty})(f)|| \\ \leq & ||(\mathrm{AffT}\psi_{m,\infty} \circ \bar{\xi}_1)(\pi(f)) - (\xi \circ \mathrm{AffT}\phi_{1,\infty} \circ \tilde{\pi})(\pi(f))|| + ||\tilde{\pi} \circ \pi(f) - f|| \\ \leq & \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1. \end{aligned}$$

Notice that $\mathrm{K}_0(A_1) = \{(\bar{k}_0^1, \bar{k}_1^1, \bar{k}_0^2, \bar{k}_1^2) : \ \sum k_{0_i}^1 = \sum k_{1_i}^1, \ \sum k_{0_i}^2 = \sum k_{1_i}^2\} \subset \mathrm{AffT}A_1$ and that $\xi_1(E) = \alpha'(E), \ \forall E \in G$, so the map ξ_1 is compatible with the map $\mathrm{K}_0\psi_{1,m} \circ \alpha_1$, that is, $\xi_1|_{\mathrm{K}_0(A_1)} = \alpha' = \mathrm{K}_0\psi_{1,m} \circ \alpha_1$.

Finally, set $\varepsilon_1 = \varepsilon/2$ to complete the proof.

□

Lemma 1.2.3. *(see [31] Theorem 3.7) Let A be a splitting interval algebra. Then for any finite set $F \subseteq A$, and $\varepsilon > 0$, there is a constant $N \in \mathbb{N}$, such that for any compatible pair $(\kappa; \theta)$ for $(A; B)$, where B is also a splitting interval algebra, there is a homomorphism $\phi : A \to B$ of the standard form which induces κ and almost induces θ in the sense that:*

$$||\mathrm{T}\phi(\tau)(f) - \theta(\tau)(f)|| < \varepsilon + \frac{2 + r_0 + r_1}{m} \cdot N \cdot ||f||, \quad \forall f \in F, \quad \tau \in TB,$$

where r_0 and r_1 are defined as in the definition 1.1 and m is the matrix size of B, i.e. $B \subseteq M_m(C[0, 1])$.

Remark 1.2.4. By the proof of Theorem 3.7 in [31], we know the following fact is true:

Let A, B be two splitting interval algebras. Then for any finite set $F \subseteq \mathrm{AffT}A$, and $\varepsilon > 0$, there is a constant $N \in \mathbb{N}$, such that for any compatible pair $(\kappa;\theta)$ for $(A;B)$, (suppose $\xi : \mathrm{AffT}A \to \mathrm{AffT}B$ is induced by θ), there is a homomorphism $\phi : A \to B$ of the standard form which induces κ and almost induces ξ in the sense that

$$||\mathrm{AffT}\phi(f) - \xi(f)|| < \varepsilon + \frac{2 + r_0 + r_1}{m} \cdot N \cdot ||f||, \quad \forall f \in F.$$

Now we deduce some properties of $*-$homomorphisms between two splitting interval algebras. Let $\phi : A \to B$ be a $*-$homomorphism with $A \subseteq \mathrm{M}_n(\mathrm{C}(X))$ and $B \subseteq \mathrm{M}_m(\mathrm{C}(Y))$ being splitting interval algebras, where $X = Y = [0, 1]$. Suppose A has r_0 and r_1 fractional endpoints at 0 and 1 respectively.

We know that for each $y \in Y$, which is not fractional, there is a set $\mathrm{SP}\phi_y$(where we count multiplicities) and a unitary u_y, such that

$$\phi(f)(y) = u_y(\bigoplus_{x \in \mathrm{SP}\phi_y} f(x))u_y^*, \quad \text{for } \forall f \in A.$$

We now define a set $\mathrm{RF}_y \subseteq \mathrm{SP}\phi_y$ (counting multiplicities) in the following manner:
Let $I_y = \{x \in \mathrm{SP}\phi_y : x \text{ is a fractional point}\}$, where we count multiplicities.
(1) If $0_1, 0_2, \cdots, 0_{r_0} \in I_y$, then we group them together to be a total point 0.

$$\text{Let new}I_y = I_y \setminus \{0_1, 0_2, \cdots, 0_{r_0}\}, \text{ (still denoting it by } I_y).$$

(2) If $1_1, 1_2, \cdots, 1_{r_1} \in I_y$, then we group them together to be a total point 1.

$$\text{Let new}I_y = I_y \setminus \{1_1, 1_2, \cdots, 1_{r_1}\}, \text{ (still denoting it by } I_y).$$

Repeat steps (1) and (2) until I_y no longer changes. Let

$$\mathrm{RF}\phi_y = I_y,$$

which we call the **Real Fractional** part of $\mathrm{SP}\phi_y$.
(**Note: By identifying the zero-groups $\{0_1, 0_2, \cdots, 0_{r_0}\}$ with 0's and the one-groups $\{1_1, 1_2, \cdots, 1_{r_1}\}$ with 1's, $\mathrm{SP}\phi_y \setminus \mathrm{RF}\phi_y$ doesn't contain fractional points**)

In what follows, we will show that the Real Fractional part $\mathrm{RF}\phi_y$ is independent of y.

Theorem 1.2.5. *Let $\phi, \psi : A \to B$ be two $*$-homomorphisms where $A \subseteq M_n(C([0, 1]))$ is a splitting interval algebra and $B \subseteq M_m(\mathbb{C})$. Suppose A has r_0 and r_1 fractional endpoints at 0 and 1 respectively. If $\mathrm{K}_0\phi = \mathrm{K}_0\psi$, then*

$$\mathrm{RF}\phi = \mathrm{RF}\psi.$$

In particular, if $\phi : A \to B$ is a homomorphism where $B \subseteq M_m(C([0,1]))$ is a splitting interval algebra, then for any $y, y' \in [0,1]$, $\mathrm{RF}\phi_y = \mathrm{RF}\phi_{y'}$.

Proof. The idea of the proof is based on K-theory. Take rank 1 projections $P_{ij} \in A \quad (i = 1, 2, \cdots, r_0, \quad j = 1, 2, \cdots, r_1)$ such that

$$P_{ij}|_{0_k} = 0 \quad \text{for} \quad k \neq i, \quad P_{ij}|_{0_i} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}_{n_{0_i} \times n_{0_i}} \quad \text{and}$$

$$P_{ij}|_{1_s} = 0 \quad \text{for} \quad s \neq j, \quad P_{ij}|_{1_j} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}_{n_{1_j} \times n_{1_j}}.$$

Then for each i, j, $\phi(P_{ij})$ and $\psi(P_{ij})$ are projections in B. Since

$$\mathrm{K}_0\phi([P_{ij}]) = \mathrm{rank}\phi(P_{ij}), \quad \mathrm{K}_0\psi([P_{ij}]) = \mathrm{rank}\psi(P_{ij})$$

and $\mathrm{K}_0\phi = \mathrm{K}_0\psi$, we have

$$\sharp\{0_i \cap \mathrm{RF}\phi\} + \sharp\{1_j \cap \mathrm{RF}\phi\} + \sharp\{\mathrm{SP}\phi \setminus \mathrm{RF}\phi\} = \sharp\{0_i \cap \mathrm{RF}\psi\} + \sharp\{1_j \cap \mathrm{RF}\psi\} + \sharp\{\mathrm{SP}\psi \setminus \mathrm{RF}\psi\},$$

for any i, j, where we count multiplicities. (1)

By the definition of $\mathrm{RF}\phi$, there exist i_0, j_0 such that $0_{i_0} \notin \mathrm{RF}\phi$ and $1_{j_0} \notin \mathrm{RF}\phi$. Therefore, by $\mathrm{K}_0\phi([P_{i_0j_0}]) = \mathrm{K}_0\psi([P_{i_0j_0}])$, we have

$$\sharp\{\mathrm{SP}\phi \setminus \mathrm{RF}\phi\} = \sharp\{0_{i_0} \cap \mathrm{RF}\psi\} + \sharp\{1_{j_0} \cap \mathrm{RF}\psi\} + \sharp\{\mathrm{SP}\psi \setminus \mathrm{RF}\psi\} \geq \sharp\{\mathrm{SP}\psi \setminus \mathrm{RF}\psi\}. \quad (2)$$

Similarly, we can get

$$\sharp\{\mathrm{SP}\psi \setminus \mathrm{RF}\psi\} \geq \sharp\{\mathrm{SP}\phi \setminus \mathrm{RF}\phi\}.$$

This yields $\sharp\{\mathrm{SP}\phi \setminus \mathrm{RF}\phi\} = \sharp\{\mathrm{SP}\psi \setminus \mathrm{RF}\psi\}$, and consequently $0_{i_0} \notin \mathrm{RF}\psi$, $1_{j_0} \notin \mathrm{RF}\psi$.

Then by (1) we get

$$\sharp\{0_i \cap \mathrm{RF}\phi\} + \sharp\{1_j \cap \mathrm{RF}\phi\} = \sharp\{0_i \cap \mathrm{RF}\psi\} + \sharp\{1_j \cap \mathrm{RF}\psi\}, \quad \text{for any } i, j. \quad (3)$$

By (2) we can also get $x_s \notin \mathrm{RF}\phi$ if and only if $x_s \notin \mathrm{RF}\psi$ for $x = 0, 1$, where s is an integer.

If $0_k \in \mathrm{RF}\phi$ for some integer k, then considering the projection P_{kj_0}, by (3), (recall $1_{j_0} \notin \mathrm{RF}\phi$ and $1_{j_0} \notin \mathrm{RF}\psi$) we obtain

$$\sharp\{0_k \cap \mathrm{RF}\phi\} = \sharp\{0_k \cap \mathrm{RF}\psi\}.$$

If $1_l \in \mathrm{RF}\phi$ for some integer l, then considering the projection $P_{i_0 l}$, by (3) we have

$$\sharp\{1_l \cap \mathrm{RF}\phi\} = \sharp\{1_l \cap \mathrm{RF}\psi\}.$$

That is $\mathrm{RF}\phi = \mathrm{RF}\psi$ (counting multiplicities).

The second statement is obvious since ϕ_y ($\phi_y(f) = \phi(f)(y)$) is homotopic to $\phi_{y'}$ for any $y, y' \in [0,1]$ and y, y' not fractional points. Thus the set $\mathrm{RF}\phi_y$ is exactly the same for any y. So we can denote $\mathrm{RF}\phi_y$ by $\mathrm{RF}\phi$.

$\square$

Remark 1.2.6. Let $\phi : A \to B$ be a homomorphism, where A, B are splitting interval algebras. Then by Theorem 1.1.12 and Theorem 1.2.5, up to a permutation we can assume that

$$\phi(f)(y) = U_y \begin{pmatrix} f(r_1(y)) & & & & & \\ & \ddots & & & & \\ & & f(r_t(y)) & & & \\ & & & f(s_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(s_l(y)) \end{pmatrix} U_y^*$$

where each $r_i(y)$ is a constant function with a value in $\mathrm{RF}\phi$, $i = 1, 2, \cdots, t$, and $s_j(y) \subseteq [0,1]$ is not fractional for $j = 1, 2, \cdots, l$. We may therefore assume $\phi = \phi_1 + \phi_2$, where

$$\phi_1(f)(y) = U_y \begin{pmatrix} f(r_1(y)) & & & & & \\ & \ddots & & & & \\ & & f(r_t(y)) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} U_y^*,$$

$$\phi_2(f)(y) = U_y \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & f(s_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(s_l(y)) \end{pmatrix} U_y^*.$$

This decomposition will be used for proving the Uniqueness Theorem in the next section.

Definition 1.2.7. *Let $\phi : A \to B$ be a homomorphism where A and B are splitting interval algebras. Denote $\mathrm{SP}\phi_y \setminus RF\phi_y = \{x_1(y), \cdots, x_l(y)\}$. Define*

$$\mathrm{SPV}(\phi) = \max_{y', y \in [0,1]} \min_{\sigma} \max_i \{|x_i(y) - x_{\sigma(i)}(y')|\},$$

where σ is any permutation.

Lemma 1.2.8. *Let A, B be two splitting interval algebras, and let $F \subseteq A$ be a finite set. Then for any integer $N > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\phi : A \to PBP$ is a unital homomorphism (where P is a projection in B) and satisfies* $\mathrm{SPV}(\phi) < \delta$, *then one of the following two statements is true:*

(a) $\mathrm{rank}(P) \geq N$.

(b) There exist projections $P_1, P_2, \cdots, P_n \in PBP$ with $\sum_i 1_{k_i} \otimes P_i = P$ such that

$$||\phi(f) - \sum_{i=1}^{n} f(x_i) \otimes P_i|| < \varepsilon \ \text{ for all } f \in F,$$

where k_i is the order of the matrix $f(x_i)$ for $i = 1, \cdots, n$.

Proof. First, we will denote by $\overline{B_a(x)}$ the closed ball in $[0,1]$ with center at x and radius a. Since F is a finite set, there is a $\delta > 0$ such that $d(x, x') \leq 2N\delta$ ($x, x' \in [0,1]$ not fractional) implies $||f(x) - f(x')|| < \varepsilon$ for all $f \in F$. Suppose that $\mathrm{rank}(P) = N_1 < N$ (i.e., (a) does not hold) and $\mathrm{SPV}(\phi) < \delta$, we shall prove that (b) of the lemma holds.

Let $y_0 \in Y$ denote the base point. Set $\mathrm{SP}(\phi_{y_0}) = \{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_s\} \cup \mathrm{RF}\phi$, where $\tilde{x}_1, \cdots, \tilde{x}_s \in [0,1]$ are not fractional, and set

$$\tilde{X} = \overline{B_\delta(\tilde{x}_1)} \cup \cdots \cup \overline{B_\delta(\tilde{x}_s)}.$$

If $0 \notin \tilde{X}$, redefine $\tilde{X}$ as

$$\tilde{X} = \tilde{X} \cup \{y \in \mathrm{RF}\phi : y \text{ is a fractional endpoint at } 0\}.$$

If $1 \notin \tilde{X}$, redefine $\tilde{X}$ as

$$\tilde{X} = \tilde{X} \cup \{y \in \mathrm{RF}\phi : y \text{ is a fractional endpoint at } 1\}.$$

Since $\mathrm{SPV}(\phi) < \delta$, by Theorem 1.2.5, it is easy to see that $\mathrm{SP}\phi_y \subseteq \tilde{X}$ for all y. Consequently, $\phi : A \to PBP$ can be factored as

$$A \xrightarrow{\text{restriction}} A|_{\tilde{X}} \xrightarrow{\tilde{\phi}} PBP.$$

On the other hand, it is easy to see that $\tilde{X}$ can be written as the disjoint union of path connected components $X_1 \cup X_2 \cup \cdots \cup X_n$ (with $n \leq N_1$) with $\mathrm{diameter}(X_i) \leq 2N\delta$. (Note that sets such as $\{0_1\}$, $\{0_2\}$ are different path connected components when $\widetilde{X} \cap (0, \varepsilon) = \emptyset$ for some $\varepsilon > 0$). As a result, $\tilde{\phi}$ can be considered as

$$A|_{X_1} \oplus A|_{X_2} \oplus \cdots \oplus A|_{X_n} \xrightarrow{\tilde{\phi}} PBP\,.$$

Set $\tilde{P}_i = \tilde{\phi}(\underbrace{0, \cdots, 0, 1}_{i}, 0, \cdots, 0)$, where 0 and 1 stand for the zero matrix and the identity matrix with corresponding sizes. Then $\sum_{i=1}^{n} \tilde{P}_i = P$. Set

$$\tilde{\phi}_i = \tilde{\phi}|_{A|_{X_i}} : A|_{X_i} \to \tilde{P}_i B \tilde{P}_i,$$

where $A|_{X_i}$ is still a splitting interval algebra.

Now we can see that X_i must be one of the following cases:

(1)$X_i = \{x\}$, for some fractional point x;

(2)$X_i \subseteq (0,1)$ and since diameter$(X_i) \leq N\delta$, there is a ball $B_i = \overline{B_{N\delta}(x_i)} \supseteq X_i$;

(3)$0 \in X_i$ (or $1 \in X_i$) with $B_i = \overline{B_{2N\delta}(0)} \supseteq X_i$ (or $B_i = \overline{B_{2N\delta}(1)} \supseteq X_i$).

In what follows, we will discuss each of these cases in detail.

If $X_i = \{x_i\}$ satisfies case (1), then there exist projections $P_i, \widetilde{P}_i \in B$ with $1_{n(f(x_i))} \otimes P_i = \tilde{P}_i$ such that $\widetilde{P}_i B \widetilde{P}_i$ can be identified with $\mathrm{M}_{n(f(x_i))} \otimes P_i B P_i$ and $\tilde{\phi}_i(f) = f(x_i) \otimes P_i$, where $n(f(x_i))$ is the size of the matrix $f(x_i)$.

If $X_j \subseteq (0,1)$ satisfies case (2), then there exist $x_j \in X_j$ and orthogonal projections $\widetilde{P}_{jk} \in B$ with $\sum_k \widetilde{P}_{jk} = \widetilde{P}_j$ and orthogonal projections $P_{jk} \in B$ with identification $\widetilde{P}_{jk} B \widetilde{P}_{jk} \cong \mathrm{M}_{n(f(x_j))} \otimes P_{jk} B P_{jk}$, (where $n(f(x_j))$ is the size of the matrix $f(x_j)$) such that $\overline{B_{N\delta}(x_j)} \supseteq X_j$ and $\tilde{\phi}_j(f)(y) = \sum_k f(\theta_k) \otimes P_{jk}$ where $\theta_k \in \mathrm{SP}\phi_y \cap X_j$. Then

$$||\tilde{\phi}_j(f) - \sum_k f(x_j) \otimes P_{jk}|| = ||\sum_k [f(\theta_k) - f(x_j)] \otimes P_{jk}|| < \varepsilon.$$

Let $P_j = \sum_k P_{jk}$. Then $||\tilde{\phi}_j(f) - f(x_j) \otimes P_j|| < \varepsilon$, and $1_{n(f(x_j))} \otimes P_j = \tilde{P}_j$.

If $0 \in X_l$ or $1 \in X_l$ (we assume $0 \in X_l$), the spectrum of $\widetilde{\phi}_l$ may have both fractional points and non-fractional points. Then the ball $\overline{B_{N\delta}(0)} \supseteq X_l$ and there are orthogonal projections $\{\widetilde{P}_{lk}\}$ and $\{\widetilde{P}_{rk}\}$ with $\sum_k \widetilde{P}_{lk} + \sum_k \widetilde{P}_{rk} = \widetilde{P}_l$ and orthogonal projections $\{P_{lk}\}$ and $\{P_{rk}\}$ with identifications $\widetilde{P}_{lk} B \widetilde{P}_{lk} \cong \mathrm{M}_{n(f(0))} \otimes P_{lk} B P_{lk}$ and $\widetilde{P}_{rk} B \widetilde{P}_{rk} \cong \mathrm{M}_{n(f(\beta_{rk}))} \otimes P_{rk} B P_{rk}$ such that

$$\tilde{\phi}_l(f)(y) = \sum_k f(\theta_{lk}) \otimes P_{lk} + \sum_k f(\beta_{rk}) \otimes P_{rk},$$

where $\theta_{lk} \in \mathrm{SP}\phi_y \cap X_l$ are not fractional and $\beta_{rk} \in \mathrm{SP}\phi_y \cap X_l$ are fractional points, $n(f(\cdot))$ is the size of $f(\cdot)$. Then

$$||\tilde{\phi}_l(f) - \sum_k f(0) \otimes P_{lk} - \sum_k f(\beta_{rk}) \otimes P_{rk}|| = ||\sum_k [f(\theta_{lk}) - f(0)] \otimes P_{lk} + 0|| < \varepsilon.$$

Let $\sum_k P_{lk} = P_l$, then $||\tilde{\phi}_l(f) - f(0) \otimes P_l - \sum_k f(\beta_{rk}) \otimes P_{rk}|| < \varepsilon$ and

$$1_{n(f(0))} \otimes P_l + \sum_k 1_{n(f(\beta_{rk}))} \otimes P_{rk} = \tilde{P}_l.$$

Based on the above discussion, we get the results.

□

Lemma 1.2.9. *Let* $A = \lim\limits_{\to}(A_n, \phi_{n,m})$ *with* $A_n = \bigoplus\limits_{i=1}^{\theta_n} P_{n,i} A_n^i P_{n,i}$ *where* $A_n^i \subseteq M_{[n,i]}(C(X_{n,i}))$ *are splitting interval algebras* ($X_{n,i} = [0,1]$) *and* $P_{n,i}$ *are non-zero projections in* A_n^i. *Further, let* $\mathcal{S} = \bigsqcup\limits_{i=1}^{\theta_n}(\mathrm{SP}(A_n^i) \setminus X_{n,i})$ *(i.e. all the fractional parts). Then for any fixed* n *and fixed* $F = \bar{F} \subset U = \overset{\circ}{U} \subset \bigsqcup\limits_{i=1}^{\theta_n} X_{n,i}$, *there exists* $m_0 > n$ *such that for any* $m \geq m_0$, *any partial map* $\phi_{n,m}^j : A_n \to A_m^j$ *satisfies either*

$$\mathrm{SP}(\phi_{n,m}^j)_y \cap (F \cup G) = \emptyset, \quad \textit{for all } y \in X_{m,j} \quad (1)$$

or

$$\mathrm{SP}(\phi_{n,m}^j)_y \cap (U \cup G) \neq \emptyset, \quad \textit{for all } y \in X_{m,j}. \quad (2)$$

for any subset G *of* $\mathcal{S}$.

Here we should emphasize that for any subset $W \subseteq [0,1]$, $0 \in W$ implies $0_i \in W$ for all i and also $1 \in W$ implies $1_j \in W$ for all j.

Proof. Pasnicu has proved a similar theorem for AH algebras (see Lemma 2.8 in [44]).

We may suppose that $n = 1$ and that $F \subset U \subset X_{1,1}$. First, we prove the case where G is an empty set.

When $F \subseteq (0,1)$, we can assume that U satisfies $\bar{U} \subseteq (0,1)$, since otherwise we can shrink U, denoted by V, that is $F \subset V = \overset{\circ}{V} \subseteq U \subseteq [0,1]$ with $\bar{V} \subseteq (0,1)$ (if (F,V) satisfies the statement, so does (F,U)). In this case, we can use the same test functions as in [44] Lemma 2.8 and all the proof of Lemma 2.8 of [44] goes through and we get: there exists $m_0 > 1$ such that for any $m \geq m_0$ and any partial map $\phi_{1,m}^j : A_1 \to A_m^j$ satisfies either (1) or (2).

Similar arguments hold for other cases, that is F may contain 0 or 1. In this case, we can also shrink U such that U satisfies: $0 \in F$ iff $0 \in U$ and $1 \in F$ iff $1 \in U$. Then using the same test functions as in Lemma 2.8 of [44], the proof of Lemma 2.8 of [44] goes through.

Summarizing the above, for $G = \emptyset$, we obtain the conclusion that is there exists $m_0 > 1$ such that for any $m \geq m_0$, any partial map $\phi_{1,m}^j : A_1 \to A_m^j$ satisfies either

$$\mathrm{SP}(\phi_{n,m}^j)_y \cap F = \emptyset, \quad \text{for all } y \in X_{m,j}$$

or

$$\mathrm{SP}(\phi_{n,m}^j)_y \cap U \neq \emptyset, \quad \text{for all } y \in X_{m,j}.$$

Now let G be any subset of $\mathcal{S}$. We still assume that U is shrunk as above, that is $0 \in F$ iff $0 \in U$ and that $1 \in F$ iff $1 \in U$. If $G \cap \mathrm{RF}\phi^j_{n,m} = \emptyset$, then

$$\mathrm{SP}(\phi^j_{n,m})_y \cap (F \cup G) = \mathrm{SP}(\phi^j_{n,m})_y \cap F,$$

$$\mathrm{SP}(\phi^j_{n,m})_y \cap (U \cup G) = \mathrm{SP}(\phi^j_{n,m})_y \cap U,$$

and the conclusion follows from the above case.

If $G \cap \mathrm{RF}\phi^j_{n,m} \neq \emptyset$, then

$$\mathrm{SP}(\phi^j_{n,m})_y \cap (U \cup G) \neq \emptyset, \quad \text{for all } y \in X_{m,j}.$$

Therefore, the conclusion of the lemma is true.

□

Using Lemma 1.2.9, and the method of proof of [44] Lemma 2.9 we can get the following corollary.

Corollary 1.2.10. *Let $A = \lim_{\to}(A_n, \phi_{n,m})$ be as in Lemma 1.2.9. Then for any fixed n, i and $\delta > 0$, there is an $m_0 > n$ such that the following is true:*

For any $F = \bar{F} \subset X_{n,i}$, and any $m \geq m_0$, we have that any partial map $\phi^{i,j}_{n,m}$ satisfies either

$$\mathrm{SP}(\phi^{i,j}_{n,m})_y \cap (F \cup G) = \emptyset, \quad \textit{for all } y \in X_{m,j}$$

or

$$\mathrm{SP}(\phi^{i,j}_{n,m})_y \cap (B_\delta(F) \cup G) \neq \emptyset, \quad \textit{for all } y \in X_{m,j}$$

for any subset G of $\mathcal{S}$ (defined in 1.2.9).

Proceeding in a way similar to the proof of Lemma 1.2.8, by using Lemma 1.2.9 and Corollary 1.2.10, we have the following lemma (cf. [44] Theorem 2.10).

Lemma 1.2.11. *Let $A = \lim_{\to}(A_n, \phi_{n,m})$ be as in Lemma 1.2.9. Then for any n, any finite subset $F^i_n \subset A^i_n \subset A_n$, any positive integer N and any $\varepsilon > 0$, there is an $m_0 > n$ such that any partial map $\phi^{i,j}_{n,m}$ with $m \geq m_0$ satisfies either*
(a) $\mathrm{rank}(\phi^{i,j}_{n,m}(P_{n,i})) \geq N \cdot \mathrm{rank}(P_{n,i})$, *or*
(b) there exists a homomorphism $\psi^{i,j}_{n,m}$ with finite dimensional range such that

$$\phi^{i,j}_{n,m}(P_{n,i}) = \psi^{i,j}_{n,m}(P_{n,i}), \textit{ and } ||\phi^{i,j}_{n,m}(f) - \psi^{i,j}_{n,m}(f)|| < \varepsilon, \quad \textit{for all } f \in F^i_n,$$

and $\mathrm{K}_0\phi^{i,j}_{n,m} = \mathrm{K}_0\psi^{i,j}_{n,m}$.

Proof. Choose $\delta > 0$ as in the proof of Lemma 1.2.8, that is $x, x' \in X_{n,i}$ with $d(x,x') \leq 2N\delta$ implies $||f(x) - f(x')|| < \varepsilon$, $\forall f \in F_n^i$ (where $d(\cdot,\cdot)$ is the canonical metric on $X_{n,i}$). Suppose that $\operatorname{rank}(\phi_{n,m}^{i,j}(P_{n,i})) < N \cdot \operatorname{rank}(P_{n,i})$.

Let $m_0 > n$ be the number obtained by applying Corollary 1.2.10 for n, i and $\delta > 0$. For any fixed $m \geq m_0$ and any fixed $y_j \in X_{m,j}$ let $G = X_{n,i} \setminus B_\delta(\mathrm{SP}(\phi_{n,m}^{i,j})_{y_j})$, where $B_\delta(\mathrm{SP}(\phi_{n,m}^{i,j})_{y_j}) = \{x \in X_{n,i} : d(x, \mathrm{SP}(\phi_{n,m}^{i,j})_{y_j}) < \delta\}$ is an open ball. Obviously G is a closed set. By Corollary 1.2.10 it follows that either

$$\text{(i)} \quad \mathrm{SP}(\phi_{n,m}^{i,j})_y \cap G = \emptyset \quad \text{for all } y \in X_{m,j}$$

or

$$\text{(ii)} \quad \mathrm{SP}(\phi_{n,m}^{i,j})_y \cap B_\delta(G) \neq \emptyset \quad \text{for all } y \in X_{m,j}.$$

However, since for $y = y_j$ the condition (ii) is false, the condition (i) holds, which implies

$$\mathrm{SP}(\phi_{n,m}^{i,j})_y \subset B_\delta(\mathrm{SP}(\phi_{n,m}^{i,j})_{y_j}) \quad \text{for all } y \in X_{m,j}.$$

i.e. $\mathrm{SPV}(\phi_{n,m}^{i,j}) < 2\delta$. The proof now continues as in the proof of Lemma 1.2.8.

□

Remark 1.2.12. By Lemma 1.2.11, we know that the following result is also true:

Let $A = \lim\limits_{\rightarrow}(A_n, \phi_{n,m})$ be as in Lemma 1.2.9. Then for any n, any finite subset $F_n^i \subset \mathrm{AffT}A_n^i$, any positive integer N and any $\varepsilon > 0$, there is an $m_0 > n$ such that any partial map $\phi_{n,m}^{i,j}$ with $m \geq m_0$ satisfies either
(a) $\operatorname{rank}(\phi_{n,m}^{i,j}(P_{n,i})) \geq N \cdot \operatorname{rank}(P_{n,i})$, or
(b) there exists a homomorphism $\psi_{n,m}^{i,j}$ with finite dimensional range—such that

$$\mathrm{K}_0\phi_{n,m}^{i,j} = \mathrm{K}_0\psi_{n,m}^{i,j}, \quad \phi_{n,m}^{i,j}(P_{n,i}) = \psi_{n,m}^{i,j}(P_{n,i}), \text{ and}$$

$$||\mathrm{AffT}\phi_{n,m}^{i,j}(f) - \mathrm{AffT}\psi_{n,m}^{i,j}(f)|| < \varepsilon, \quad \text{for all } f \in F_n^i.$$

Lemma 1.2.13. *Let A_1, A_2, A_3 be splitting interval algebras, and let $\xi : \mathrm{AffT}A_2 \to \mathrm{AffT}A_3$ be a unital positive linear map which is compatible with $\tilde{\Lambda}$, where $\tilde{\Lambda} : A_2 \to A_3$ is a homomorphism. Let $\phi : A_1 \to A_2$ be a unital homomorphism, $\varepsilon > 0$, and let $E \subseteq A_1$ be a finite set. If there is a unital homomorphism $\psi : A_1 \to A_2$ defined by evaluation at points with $\mathrm{K}_0\phi = \mathrm{K}_0\psi$ and*

$$||\mathrm{AffT}\phi(f) - \mathrm{AffT}\psi(f)|| < \varepsilon, \ \forall f \in E,$$

then there is a homomorphism $\Lambda : A_1 \to A_3$ such that
(1) $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\tilde{\Lambda}) \circ \mathrm{K}_0(\phi)$ and $\mathrm{AffT}\Lambda(f) = \xi \circ \mathrm{AffT}\psi(f)$, $\forall f \in E$,
(2) $||\mathrm{AffT}\Lambda(f) - \xi \circ \mathrm{AffT}\phi(f)|| < \varepsilon$, $\forall f \in E$.
Here ψ is defined by evaluation at points, which means that there exist $x_1, \cdots, x_n \in \mathrm{SP}(A_1)$, projections $p_1, \cdots, p_n$ such that $\psi(f) = \sum_{i=1}^n f(x_i) \otimes p_i$, where p_i, p_j are orthogonal, $\sum_{i=1}^n 1_{k_i} \otimes p_i = 1_{A_2}$, and k_i is the size of the matrix $f(x_i)$.

Proof. Let $\theta : \mathrm{T}A_3 \to \mathrm{T}A_2$ such that $\theta^* = \xi$, and $\Lambda = \tilde{\Lambda} \circ \psi$. Then $\mathrm{T}\Lambda = \mathrm{T}\psi \circ \mathrm{T}\tilde{\Lambda}$, and for any $\tau \in \mathrm{T}A_3$, $f \in E$,

$$\begin{aligned}\theta(\tau)(f(x_i) \otimes p_i) &= \theta(\tau)(f(x_i)) \cdot \theta(\tau)(p_i) = \theta(\tau)(\sum_j f_{jj}(x_i)e_{jj})\tau(\tilde{\Lambda}(p_i)) \\ &= \sum_j f_{jj}(x_i)\theta(\tau)(e_{jj})\tau(\tilde{\Lambda}(p_i)) = \sum_j f_{jj}(x_i)\tau(\tilde{\Lambda}(e_{jj}))\tau(\tilde{\Lambda}(p_i)).\end{aligned}$$

This is because θ and $\tilde{\Lambda}$ are compatible, where $f_{jj}(x_i)$ is the value at the (j, j) position of $f(x_i)$ and e_{jj} is the matrix with 1 at the (j, j) position and 0 at others, which is a projection. Then

$$\begin{aligned}\theta(\tau)(f(x_i) \otimes p_i) &= \tau(\tilde{\Lambda}(\sum_j f_{jj}(x_i)e_{jj}))\tau(\tilde{\Lambda}(p_i)) \\ &= \tau(\tilde{\Lambda}(f(x_i))\tau(\tilde{\Lambda}(p_i)) \\ &= \tau(\tilde{\Lambda}(f(x_i) \otimes p_i)).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathrm{T}\psi \circ \theta(\tau)(f) &= \theta(\tau)(\psi(f)) \\ &= \theta(\tau)(\sum_i f(x_i) \otimes p_i) \\ &= \sum_i \theta(\tau)(f(x_i) \otimes p_i) \\ &= \sum_i \tau(\tilde{\Lambda}(f(x_i) \otimes p_i)) \\ &= \tau(\tilde{\Lambda}(\psi(f))) \\ &= \tau(\Lambda(f)) = \mathrm{T}\Lambda(\tau)(f), \text{ for all } f \in E, \tau \in \mathrm{T}A_3, \qquad (1)\end{aligned}$$

which means

$$\mathrm{Aff}\mathrm{T}\Lambda(f) = \xi \circ \mathrm{Aff}\mathrm{T}\psi(f), \text{ for all } f \in E.$$

In addition, $||\mathrm{Aff}\mathrm{T}\phi(f) - \mathrm{Aff}\mathrm{T}\psi(f)|| < \varepsilon$ (for all $f \in E$) implies

$$||\mathrm{T}\psi(\tau)(f) - \mathrm{T}\phi(\tau)(f)|| < \varepsilon, \ \forall f \in E, \ \tau \in \mathrm{T}A_2. \qquad (2)$$

Then by (1) and (2),

$$||\mathrm{T}\Lambda(\tau)(f) - \mathrm{T}\phi \circ \theta(\tau)(f)|| = ||\mathrm{T}\psi \circ \theta(\tau)(f) - \mathrm{T}\phi \circ \theta(\tau)(f)|| < \varepsilon, \ \forall f \in E, \ \tau \in \mathrm{T}A_3,$$

which implies

$$||\mathrm{Aff}\mathrm{T}\Lambda(f) - \xi \circ \mathrm{Aff}\mathrm{T}\phi(f)|| < \varepsilon, \text{ for all } f \in E.$$

Since $\mathrm{K}_0\phi = \mathrm{K}_0\psi$, we have $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\tilde{\Lambda} \circ \psi) = \mathrm{K}_0(\tilde{\Lambda}) \circ \mathrm{K}_0\phi$.

□

Following theorem is the main theorem of this section, and is referred to as Existence Theorem.

Theorem 1.2.14. *(Existence Theorem) Let* $A = \lim\limits_{n\to\infty}(A_n, \phi_{n,m})$*,* $B = \lim\limits_{n\to\infty}(B_n, \psi_{n,m})$ *be* C^**-algebras with the ideal property where* $A_n = \bigoplus\limits_{i=1}^{k_n} A_n^i$*,* $B_m = \bigoplus\limits_{j=1}^{l_m} B_m^j$*,* A_n^i*,* B_m^j *are splitting interval algebras, and* $\phi_{n,m}$*,* $\psi_{n,m}$ *are unital. Suppose that there exists an isomorphism* $\xi : \mathrm{AffT}A \to \mathrm{AffT}B$*, and an ordered group isomorphism* $\alpha : \mathrm{K}_0A \to \mathrm{K}_0B$ *which is compatible with* ξ*. Then for any* $\varepsilon > 0$ *and any finite set* $F \subseteq \mathrm{AffT}A_1$*, there exists a map* $\Lambda : A_1 \to B_m$ *(m large) such that*

(1) $||\mathrm{AffT}\psi_{m,\infty} \circ \mathrm{AffT}\Lambda(f) - \xi \circ \mathrm{AffT}\phi_{1,\infty}(f)|| < \varepsilon,\ \forall f \in F$*, and*

(2) $\mathrm{K}_0\Lambda = \mathrm{K}_0\psi_{1,m} \circ \alpha_1$*.*

Proof. By Lemma 1.2.1, there exists an intertwining at the K_0 stage

$$\begin{array}{ccccccccc}
\mathrm{K}_0A_1 & \longrightarrow & \mathrm{K}_0A_2 & \longrightarrow & \mathrm{K}_0A_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\
\alpha_1\big\downarrow & \overset{\beta_1}{\nearrow} & \alpha_2\big\downarrow & \overset{\beta_2}{\nearrow} & \alpha_3\big\downarrow & \overset{\beta_3}{\nearrow} & & & \alpha\big\downarrow \\
\mathrm{K}_0B_1 & \longrightarrow & \mathrm{K}_0B_2 & \longrightarrow & \mathrm{K}_0B_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B
\end{array}$$

where α_i, β_i are scaled ordered group homomorphisms and there exist homomorphisms $\tilde{\Lambda}_i : A_i \to B_i$, $\tilde{\mathcal{M}}_i : B_i \to A_{i+1}$ satisfying $\mathrm{K}_0\tilde{\Lambda}_i = \alpha_i$ and $\mathrm{K}_0\tilde{\mathcal{M}}_i = \beta_i$. Since α and ξ are compatible, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{K}_0(A) & \overset{\sigma}{\to} & \mathrm{AffT}A \\
\big\downarrow\alpha & & \big\downarrow\xi \\
\mathrm{K}_0(B) & \overset{\sigma}{\to} & \mathrm{AffT}B
\end{array}.$$

Let $F = \bigoplus\limits_{i=1}^{k_1} F_i$, where $F_i \subset \mathrm{AffT}A_1^i$ are finite sets. For each finite set F_i, any fixed $\varepsilon/2 > 0$, there exists N_i satisfying the conditions of Lemma 1.2.3.

Let $N = \max\limits_i\{N_i\}$, $K = 2^{\lfloor \frac{4}{\varepsilon} \rfloor + 1}(2 + 2n_1) \cdot N \cdot \max\{||f|| : f \in F\}$, where $n_1 = \max\limits_i\{n_1^i :\ n_1^i$ is the size of the $i-th$ block of $A_1\}$. Then for $\varepsilon/2 > 0$, $K > 0$, and the finite set F, applying Lemma 1.2.11 and Remark 1.2.12, we obtain $n_1 > 0$ such that for any $n' \geq n_1$, the partial map $\phi_{1,n'}^{i,j}$ (for any i, j) satisfies either

(a) $\mathrm{rank}(\phi_{1,n'}^{i,j}(P_{1,i})) \geq K \cdot \mathrm{rank}(P_{1,i})$, or

(b) $\phi_{1,n'}^{i,j}(P_{1,i}) = \psi_{1,n'}^{i,j}(P_{1,i})$, where $\psi_{1,n'}^{i,j}$ is a homomorphism with finite dimensional image, and

$$||\mathrm{AffT}\phi_{1,n'}^{i,j}(f) - \mathrm{AffT}\psi_{1,n'}^{i,j}(f)|| < \frac{\varepsilon}{2},\ \forall f \in F_i.$$

Here $P_{1,i}$ is the unit of A_1^i.

Fix $n' = n_1$, applying Lemma 1.2.2, we obtain an integer $m > n'$ and $\xi_n' : \mathrm{AffT}A_{n'} \to \mathrm{AffT}B_m$, such that for all $f \in F$, we have

$$||\mathrm{AffT}\psi_{m,\infty} \circ \xi_n'(\mathrm{AffT}\phi_{1,n'}(f)) - \xi \circ \mathrm{AffT}\phi_{n',\infty}(\mathrm{AffT}\phi_{1,n'}(f))|| < \frac{\varepsilon}{4}.$$

This means that the following diagram is approximately commutative within ε:

$$\begin{array}{ccccc} \mathrm{AffT}A_1 & \xrightarrow{\mathrm{AffT}\phi_{1,n'}} & \mathrm{AffT}A_{n'} & \xrightarrow{\mathrm{AffT}\phi_{n',\infty}} & \mathrm{AffT}A \\ & \searrow^{\xi_1} & \downarrow{\xi'_n} & & \downarrow{\xi} \\ & & \mathrm{AffT}B_m & \xrightarrow{\mathrm{AffT}\psi_{m,\infty}} & \mathrm{AffT}B \end{array}$$

Let $\xi_1 = \xi'_n \circ \mathrm{AffT}\phi_{1,n'} : \mathrm{AffT}A_1 \to \mathrm{AffT}B_m$. Then

$$||\mathrm{AffT}\psi_{m,\infty} \circ \xi_1(f) - \xi \circ \mathrm{AffT}\phi_{1,\infty}(f)|| < \frac{\varepsilon}{4}, \quad \forall f \in F. \qquad (*)$$

By Lemma 1.2.2, ξ'_n is compatible with $\mathrm{K}_0\psi_{n',m}\circ\alpha_{n'}$, so ξ_1 is compatible with $\mathrm{K}_0\psi_{n',m}\circ\alpha_{n'} \circ \mathrm{K}_0\phi_{1,n'} = \mathrm{K}_0\psi_{1,m} \circ \alpha_1$.

Now we have a compatible pair $(\xi_1,\ \mathrm{K}_0\psi_{1,m} \circ \alpha_1)$ between A_1 and B_m. By Section 3 of [31], we only need to consider the case where both A_1 and B_m have just one block.

(1) If $\phi^{1,j}_{1,n'}$ satisfies condition (a), then by Lemma 1.2.3 and its Remark, there exists a unital homomorphism $\Lambda : A_1 \to B_m$ such that for any $f \in F$, we have $\mathrm{K}_0\Lambda = \mathrm{K}_0\psi_{1,m} \circ \alpha_1$ and

$$||\mathrm{AffT}\Lambda(f) - \xi_1(f)|| < \frac{\varepsilon}{2} + \frac{2 + r_0 + r_1}{m} \cdot N \cdot ||f|| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{4}.$$

(2) If $\phi^{1,j}_{1,n'}$ satisfies condition (b), then by Lemma 1.2.11 and Remark 1.2.12, there exists a homomorphism $\Lambda : A_1 \to B_m$ such that $\mathrm{K}_0\Lambda = \mathrm{K}_0\psi_{n',m} \circ \alpha_{n'} \circ \mathrm{K}_0\phi_{1,n'} = \mathrm{K}_0\psi_{1,m} \circ \alpha_1$ and $||\mathrm{AffT}\Lambda(f) - \xi'_n \circ \mathrm{AffT}\phi_{1,n'}(f)|| < \frac{\varepsilon}{2}, \forall f \in F$, that is $||\mathrm{AffT}\Lambda(f) - \xi_1(f)|| < \frac{\varepsilon}{2}$.

Taking (1), (2) and $(*)$ together,

$$\begin{aligned} & ||\mathrm{AffT}\psi_{m,\infty} \circ \mathrm{AffT}\Lambda(f) - \xi \circ \mathrm{AffT}\phi_{1,\infty}(f)|| \\ = \ & ||\mathrm{AffT}\psi_{m,\infty} \circ \mathrm{AffT}\Lambda(f) - \mathrm{AffT}\psi_{m,\infty} \circ \xi_1(f) + \mathrm{AffT}\psi_{m,\infty} \circ \xi_1(f) - \xi \circ \mathrm{AffT}\phi_{1,\infty}(f)|| \\ < \ & \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

and $\mathrm{K}_0\Lambda = \mathrm{K}_0\psi_{1,m} \circ \alpha_1$, which completes the proof.

□

Remark 1.2.15. In Theorem 1.2.14, homomorphisms $\phi_{n,m}, \psi_{n,m}$ are required to be unital. When we prove the Classification Theorem in section 4 below, we will use projections to cut down algebras to make the homomorphisms unital. e.g: If $\phi : A \to B$ is a non-unital homomorphism, then $\phi : A \to \phi(1_A)B\phi(1_A)$ is a unital homomorphism.

1.3 Uniqueness Theorem

Lemma 1.3.1. *(see Lemma 2.15 of [35]). Suppose that X is a path connected compact metric space, and $\eta,\ \delta > 0$. Then there is a finite set $H \subset \mathrm{AffT}\, C(X) = C(X)$ such that the following statement is true: Let Y be a compact metric space and let two unital homomorphisms $\phi,\ \psi : C(X) \to PM_k(C(Y))P$ satisfy the following two conditions*

(1) for any $x \in X$ and the $\frac{\eta}{8}$-ball of x $B_{\eta/8}(x) = \{x' \in X,\ \mathrm{dist}(x', x) < \frac{\eta}{8}\}$,

$$\sharp(\mathrm{SP}\phi_y \cap B_{\eta/8}(x)) \geq \delta \cdot \sharp(\mathrm{SP}\phi_y),$$

for all $y \in Y$ (notice that $\sharp(\mathrm{SP}\phi_y) = \mathrm{rank}(P)$);

(2) $||\mathrm{AffT}\phi(h) - \mathrm{AffT}\psi(h)|| < \delta/4$, for all $h \in H$.

Then $\mathrm{SP}\phi_y$ and $\mathrm{SP}\psi_y$ can be paired to within η for each $y \in Y$, that is, one can write

$$\mathrm{SP}\phi_y = \{x_1, \cdots, x_n\} \text{ and } \mathrm{SP}\psi_y = \{x'_1, \cdots, x'_n\}$$

(here $n = \mathrm{rank}(P)$) such that $\mathrm{dist}(x_i, x'_i) < \eta$, for each i.

Lemma 1.3.2. *([31], Proposition 4.1). Let $\phi,\ \psi : A \to B$ be two unital standard maps between two splitting interval algebras. Suppose that:*

(1) $\phi_* = \psi_* : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$, and

(2) $\phi,\ \psi$ have the same eigenvalue maps.

Then for any finite set $F \subset A$ and $\varepsilon > 0$, there is a unitary $U \in B$ such that

$$||U\phi(f)U^* - \psi(f)|| < \varepsilon, \text{ for all } f \in F.$$

Lemma 1.3.3. *Let $A \subseteq M_n(C[0,1])$ and $B \subseteq M_m(C[0,1])$ be two splitting interval algebras. Suppose that $\phi : A \to B$ can be expressed as*

$$\phi(f)(y) = U_y \begin{pmatrix} (f(r_1))_{k_1\times k_1} & & & & & \\ & \ddots & & & & \\ & & (f(r_t))_{k_t\times k_t} & & & \\ & & & (f(s_1(y)))_{n\times n} & & \\ & & & & \ddots & \\ & & & & & (f(s_l(y)))_{n\times n} \end{pmatrix} U_y^*,$$

where $r_1, \cdots, r_t$ are constant functions among $\{0_1, \cdots, 0_{r_0}, 1_1, \cdots, 1_{r_1}\} \subseteq SP(A)$, and for each y the eigenvalues $s_i(y)$ are distinct on $(0,1)$, that is, if $s_i(y),\ s_j(y) \in (0,1)$ and $i \neq j$, then $s_i(y) \neq s_j(y)$. Let $s'_1(y), \cdots, s'_l(y)$ be a sequence of continuous functions with values in $[0,1]$ and with the property $s_j(y) = 0$ (or 1) implies $s'_j(y) =$

0 (*or* 1) *for any* j. *Let* ϕ' *be defined as*

$$\phi'(f)(y) = U_y \begin{pmatrix} (f(r_1)) & & & & & \\ & \ddots & & & & \\ & & (f(r_t)) & & & \\ & & & (f(s'_1(y))) & & \\ & & & & \ddots & \\ & & & & & (f(s'_l(y))) \end{pmatrix} U_y^*,$$

then $\phi'(A) \subseteq B$.

Proof. For any fixed $f \in A$, we will show $\phi'(f) \in B$, that is $\phi'(f)(0) \in B|_{\{0\}}$ and $\phi'(f)(1) \in B|_{\{1\}}$.

Define $g \in A$ as follows:

First let

$$g(0) = f(0), \quad g(1) = f(1) \;\; \textit{and}$$
$$g(s_i(0)) = f(s'_i(0)) \;\; \text{for } s_i(0) \neq 0,$$

then continuously connect these finite points.

Since $\{s_i(0)\}_{i=1}^{l}$ are distinct in $(0, 1)$, g is well-defined. By this definition, g satisfies the following properties:

(i) $g \in A$, since $g(0) = f(0)$ and $g(1) = f(1)$. Thus $\phi(g) \in B$.

(ii) $g(s_i(0)) = f(s'_i(0))$ for all i since $s_i(0) = 0$ (or 1) $\Rightarrow s'_i(0) = 0$ (or 1).

Then

$$\begin{aligned}
\phi'(f)(0) &= U_0 \begin{pmatrix} (f(r_1)) & & & & & \\ & \ddots & & & & \\ & & (f(r_t)) & & & \\ & & & (f(s'_1(0))) & & \\ & & & & \ddots & \\ & & & & & (f(s'_l(0))) \end{pmatrix} U_0^* \\
&= U_0 \begin{pmatrix} (g(r_1)) & & & & & \\ & \ddots & & & & \\ & & (g(r_t)) & & & \\ & & & (g(s_1(0))) & & \\ & & & & \ddots & \\ & & & & & (g(s_l(0))) \end{pmatrix} U_0^* \\
&= \phi(g)(0) \in B|_{\{0\}}.
\end{aligned}$$

Similarly we can define another function $h \in A$ satisfying properties similar to g above to deduce that $\phi'(f)(1) = \phi(h)(1) \in B|_{\{1\}}$. Then we can get $\phi'(f) \in B$.

□

Theorem 1.3.4. *(Uniqueness Theorem) Let A, B be two splitting interval algebras, where $A \subseteq M_n(C[0,1])$, $B \subseteq M_m(C[0,1])$. For any given finite set $F \subseteq A$, and $\varepsilon > 0$, there exists $\eta > 0$ such that for any $\delta > 0$, there is a finite set $H(\eta, \delta, X) \subseteq \mathrm{AffT}A$ (where $X = [0,1]$) such that the following statement is true:*

If two unital homomorphisms $\phi, \psi : A \to B$ satisfy the conditions

(1) ϕ or ψ has the property $\mathrm{sdp}(\eta, \delta)$,

(2) $||\mathrm{AffT}\phi(h) - \mathrm{AffT}\psi(h)|| < \delta$, $\forall h \in H(\eta, \delta, X)$, and

(3) $\mathrm{K}_0\phi = \mathrm{K}_0\psi$,

then there exists a unitary $U \in B$ such that $||\phi(f) - U\psi(f)U^*|| < \varepsilon$, $\forall f \in F$.

Proof. First, we decompose ϕ and ψ. By Theorem 1.1.12, Remark 1.2.6, we can assume that

$$\phi(f)(y) = U_y \begin{pmatrix} f(r_1) & & & & & \\ & \ddots & & & & \\ & & f(r_t) & & & \\ & & & f(s_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(s_l(y)) \end{pmatrix} U_y^*,$$

where $U_y \in \mathrm{M}_m(\mathrm{C}[0,1])$, $r_i \in \mathrm{RF}\phi$, $i = 1, \cdots, t$, $s_j(y) \in [0,1]$ for any y, j. For each fixed y, the eigenvalues $s_i(y)$ are distinct in $(0,1)$, that is, if $s_i(y)$, $s_j(y) \in (0,1)$ and $i \neq j$, then $s_i(y) \neq s_j(y)$.

Let

$$\phi_1(f)(y) = U_y \begin{pmatrix} f(r_1) & & & & & \\ & \ddots & & & & \\ & & f(r_t) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} U_y^*$$

and

$$\phi_2(f)(y) = U_y \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & f(s_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(s_l(y)) \end{pmatrix} U_y^*.$$

Then $\phi = \phi_1 + \phi_2$, $\mathrm{SP}\phi = \mathrm{SP}\phi_1 \cup \mathrm{SP}\phi_2$, $\mathrm{AffT}\phi = \mathrm{AffT}\phi_1 + \mathrm{AffT}\phi_2$. Similarly, $\psi = \psi_1 + \psi_2$ with corresponding unitary functions $V_y \in \mathrm{M}_m(\mathrm{C}[0,1])$ and eigenvalue maps $\widehat{r_i} \in \mathrm{RF}\psi$, $\widehat{s_j}(y) \in [0,1]$. We will show that the spectra of ϕ_1 and ψ_1, ϕ_2 and ψ_2 can be paired respectively.

Since $\mathrm{K}_0\phi = \mathrm{K}_0\psi$, by Theorem 1.2.5, we know that $\mathrm{RF}\phi = \mathrm{RF}\psi$. This means that $r_i = \widehat{r}_{\sigma(i)}$, $i = 1, \cdots, t$, where σ is a permutation. We then have $\mathrm{SP}(\phi_1) = \mathrm{SP}(\psi_1)$.

Let $H(\eta, \delta, X)$ be the finite set mentioned in Lemma 1.3.1.

Define $\widetilde{\phi}_2,\ \widetilde{\psi}_2 : \mathrm{M}_n(\mathrm{C}[0,1]) \to \mathrm{M}_m(\mathrm{C}[0,1])$ by

$$\widetilde{\phi}_2(f)(y) = U_y \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & f(s_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(s_l(y)) \end{pmatrix} U_y^*,$$

$$\widetilde{\psi}_2(f)(y) = V_y \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & f(\widehat{s}_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(\widehat{s}_l(y)) \end{pmatrix} V_y^*.$$

It is obvious that $\mathrm{SP}\phi_2 = \mathrm{SP}\widetilde{\phi}_2$, $\mathrm{SP}\psi_2 = \mathrm{SP}\widetilde{\psi}_2$ and

$$\begin{aligned} & ||\mathrm{AffT}\widetilde{\phi}_2(f) - \mathrm{AffT}\widetilde{\psi}_2(f)|| \\ = \ & ||\mathrm{AffT}\phi_2(f) - \mathrm{AffT}\psi_2(f)|| \\ = \ & a^{-1} \cdot ||\mathrm{AffT}\phi(f) - \mathrm{AffT}\psi(f)|| \\ < \ & \tfrac{\delta}{a}, \quad \forall f \in H(\eta, \delta, X), \end{aligned}$$

where $a = \frac{rank(\phi_2(1_A))}{m} < 1$.

Since ϕ or ψ has the property $\mathrm{sdp}(\eta, \delta)$ (suppose ϕ has), that is

$$\sharp(\mathrm{SP}\phi_y \cap B_\eta(x)) \geq \delta \cdot \sharp(\mathrm{SP}\phi_y),$$

then

$$\begin{aligned} \sharp(\mathrm{SP}(\widetilde{\phi}_2)_y \cap B_\eta(x)) & = \sharp(\mathrm{SP}(\phi_2)_y \cap B_\eta(x)) \\ & \geq \delta \cdot \sharp(\mathrm{SP}\phi_y) \\ & = \delta \cdot \tfrac{\sharp(\mathrm{SP}(\phi_2)_y)}{a} \\ & = \tfrac{\delta}{a} \cdot \sharp(\mathrm{SP}(\widetilde{\phi}_2)_y) \end{aligned},$$

which means $\widetilde{\phi}_2$ has the property $\mathrm{sdp}(\eta, \frac{\delta}{a})$. Consequently, $\widetilde{\phi}_2$ or $\widetilde{\psi}_2$ has the property $\mathrm{sdp}(\eta, \frac{\delta}{a})$.

By Lemma 1.3.1, $\mathrm{SP}(\widetilde{\phi}_2)_y$ and $\mathrm{SP}(\widetilde{\psi}_2)_y$ can be paired within $8\eta < 1$ for each $y \in Y$.

Since

$$\mathrm{SP}\phi_y = \mathrm{SP}(\phi_1)_y \cup \mathrm{SP}(\phi_2)_y = \mathrm{RF}(\phi) \cup \mathrm{SP}(\widetilde{\phi}_2)_y,$$
$$\mathrm{SP}\psi_y = \mathrm{SP}(\psi_1)_y \cup \mathrm{SP}(\psi_2)_y = \mathrm{RF}(\psi) \cup \mathrm{SP}(\widetilde{\psi}_2)_y,$$

$\mathrm{SP}\phi_y$ and $\mathrm{SP}\psi_y$ can be paired within 8η for each $y \in Y$. Without loss of generality, we can assume that $s_j(y)$ and $\widehat{s}_j(y)$ are the corresponding pair.

For each j, define

$$\alpha_j(y) = \min\{s_j(y),\ \widehat{s}_j(y)\},$$
$$\beta_j(y) = \max\{s_j(y),\ \widehat{s}_j(y)\}.$$

Since $s_j, \widehat{s}_j$ are continuous, there exist continuous functions $\lambda_j(y) : [0,1] \to [0,1]$ satisfying

$$\alpha_j(y) \le \lambda_j(y) \le \beta_j(y) \quad \text{for all } y, \text{ and}$$
$$\alpha_j(y) = 0 \Rightarrow \lambda_j(y) = 0, \quad \beta_j(y) = 1 \Rightarrow \lambda_j(y) = 1, \quad \text{for all } y.$$

(Notice that $\mathrm{dist}(\alpha_j(y), \beta_j(y)) < 8\eta < 1$, so there exists no y such that both $\alpha_j(y) = 0$ and $\beta_j(y) = 1$ hold.)

Then $\{\lambda_j\}_{j=1}^{l}$ is a sequence of continuous functions satisfying

$$s_j(y) = 0 \Rightarrow \lambda_j(y) = 0, \quad \text{for all } j,$$
$$s_j(y) = 1 \Rightarrow \lambda_j(y) = 1, \quad \text{for all } j.$$

Define

$$\Phi(f)(y) = U_y \begin{pmatrix} f(r_1) & & & & & \\ & \ddots & & & & \\ & & f(r_t) & & & \\ & & & f(\lambda_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(\lambda_l(y)) \end{pmatrix} U_y^*,$$

$$\Psi(f)(y) = V_y \begin{pmatrix} f(r_1) & & & & & \\ & \ddots & & & & \\ & & f(r_t) & & & \\ & & & f(\lambda_1(y)) & & \\ & & & & \ddots & \\ & & & & & f(\lambda_l(y)) \end{pmatrix} V_y^*.$$

Then by Lemma 1.3.3 (notice the properties of $\{\lambda_j(y)\}_{j=1}^{l}$), Φ, Ψ are two homomorphisms from A to B and $\mathrm{SP}\Phi_y = \mathrm{SP}\Psi_y$, for all $y \in [0,1]$. By Lemma 1.3.2, we can find a unitary element $U \in B$ such that

$$||\Phi(f) - U\Psi(f)U^*|| < \frac{\varepsilon}{2}, \quad \forall f \in F.$$

By the definition of $\{\lambda_j(y)\}_{j=1}^{l}$, we can also get the following facts:

$$\lambda_j(y) \text{ and } s_j(y) \text{ can be paired within } 8\eta \text{ for all } y, j;$$

$$\lambda_j(y) \text{ and } \widehat{s}_j(y) \text{ can be paired within } 8\eta \text{ for all } y, j.$$

Finally, take η small enough such that

$$||\Phi(f) - \phi(f)|| < \varepsilon/4; \quad ||\Psi(f) - \psi(f)|| < \varepsilon/4, \ \forall f \in F.$$

Then we have

$$||\phi(f) - U\psi(f)U^*|| < \varepsilon, \quad \forall f \in F.$$

□

Remark 1.3.5. In the last part of the proof of Theorem 1.3.4, the purpose of defining $s_j(y)'$ in that form was to preserve the special structure of the algebra at the endpoints 0 and 1. Otherwise the image of $\overline{\phi}$ and $\overline{\psi}$ might not be in B.

Remark 1.3.6. In the proof of Lemma 1.3.1 (see [35]), the finite set $H(\eta, \delta, X)$ was constructed as follows: Choose $H_1 = \{\chi_{T,8/\eta} : \mathrm{T} \subseteq X \text{ is a closed set}\}$. Since H_1 is a family of equally continuous functions, there is a finite set $H \subset H_1$ such that $\mathrm{dist}(h, H_1) < \delta/8$ for all $h \in H$, which we denote by $H(\eta, \delta, X)$.

The set $H(\eta, \delta, X)$ in Lemma 1.3.1 is in $\mathrm{C}(X)$, however, it can also be seen as belonging to $\mathrm{AffT}(A)$ by defining $f(0_i) = f(0)$ for all i and $f(1_j) = f(1)$ for all j.

Notice that for any connected closed subset X' of X, the finite set $H(\eta, \delta, X')$ is equal to $\{f|_{X'} : f \in H(\eta, \delta, X)\} = \pi(H(\eta, \delta, X))$, where $\pi(f) = f|_{X'}, \forall f \in H(\eta, \delta, X)$. Thus, we have the following corollary.

Corollary 1.3.7. *Let A, B be two splitting interval algebras. For a given finite set $F \subset A$, any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $\delta > 0$, there is a finite set $H(\eta, \delta, X)$ (X=[0,1]), which makes the following statement true*

For any connected subset $X_s \subset [0, 1]$, if two unital homomorphisms ϕ_s, ψ_s : $A|_{X_s} \to B$ satisfy the conditions:

(1) ϕ_s or ψ_s has the property $\mathrm{sdp}(\eta, \delta)$,

(2) $||\mathrm{AffT}\phi_s(h) - \mathrm{AffT}\psi_s(h)|| < \delta, \ \forall h \in H(\eta, \delta, X_s) = \pi_s(H(\eta, \delta, X))$, and

(3) $\mathrm{K}_0\phi_s = \mathrm{K}_0\psi_s$,

then there exists a unitary $U \in B$ such that

$$||\phi_s(f) - U\psi_s(f)U^*|| < \varepsilon, \ \forall f \in \pi_s(F),$$

where $\pi_s(f) = f|_{X_s}$, for all $f \in F$.

Preparation for Dichotomy Theorem.

When proving the isomorphism between C*-algebras $A = \lim\limits_{n\to\infty}(A_n, \phi_{n,m})$, $B = \lim\limits_{n\to\infty}(B_n,\, \psi_{n,m})$, it is necessary to consider whether or not the non-zero partial maps $\phi^{i,j}_{n,m}$, $\psi^{i,j}_{n,m}$ have the spectrum distribution property (sdp(η, δ); see 1.1.7). This is an important condition in the Uniqueness theorem, which is one of the key components of the intertwining argument used to prove isomorphism of inductive limit C*-algebras. Therefore, it is important to be able to ensure that the partial maps have the spectrum distribution property. This technique is inspired by [30].

For any fixed $A_n = \bigoplus\limits_{i=1}^{k_n} A_n^i$ where $A_n^i \subset \mathrm{M}_{[n,i]}(\mathrm{C}(X_{n,i}))$ are splitting interval algebras, and for any $\eta > 0$, we may apply Corollary 1.2.10 with $\delta = \eta/4$ to obtain $m_0 > n$ satisfying the conclusion of Corollary 1.2.10 for all $i = 1, 2, \cdots, k_n$. Considering the partial map $\phi^{i,j}_{n,m}$ $(m \geq m_0)$, by the first isomorphism theorem, we can demonstrate that there exists an injective map

$$\phi'^{i,j}_{n,m} : A_n^i / ker\phi^{i,j}_{n,m} \to A_m^j.$$

Let X_i^j be the closed subset of $\mathrm{SP}A_n^i$ such that, in the natural way,

$$A_n^i / ker\phi^{i,j}_{n,m} \cong A_n^i|_{X_i^j}.$$

In fact, $X_i^j = \bigcup\limits_{y\in X_{m,j}} \mathrm{SP}(\phi^{i,j}_{n,m})_y$. Here the topology of $\mathrm{SP}A_n^i$ is the natural non-Hausdorff topology. If $0 \in X_i^j$, then $0_1, \cdots 0_{r_0} \in X_i^j$; if $0 \notin X_i^j$, then there exists $0 < \delta < 1$ such that $(0, \delta) \cap X_i^j = \emptyset$. However, X_i^j may contain some fractional points of 0. A similar situation occurs at the endpoint 1.

Set $\pi_{i,j}(f) = f|_{X_i^j}$, $\pi = \bigoplus\limits_{i,j} \pi_{i,j}$. Then $\phi_{n,m}$ can be written as

$$A_n \xrightarrow{\ \pi\ } \left(B = \bigoplus_i \bigoplus_j A_n^i|_{X_i^j} \right) \xrightarrow{\ \phi\ } A_m \ ,$$

where $\phi = \bigoplus\limits_i \bigoplus\limits_j \phi'^{i,j}_{n,m}$. Notice that $X_i^j \cap (0, 1)$ may not necessarily be an finite disjoint union of finite intervals; we wish to enlarge X_i^j appropriately in order to turn it into a finite disjoint union of intervals or fractional points.

Claim: For any $x \in X_i^j$, $B_{\eta/4}(x) \cap \mathrm{SP}(\phi^{i,j}_{n,m})_y \neq \emptyset$ for all $y \in X_{m,j}$.

In fact, for any non-fractional point $x_0 \in X_i^j$, since $X_i^j = \bigcup\limits_{y\in X_{m,j}} \mathrm{SP}(\phi^{i,j}_{n,m})_y$, there exists $y_0 \in X_{m,j}$ such that $x_0 \in \mathrm{SP}(\phi^{i,j}_{n,m})_{y_0}$. Then by Corollary 1.2.10,

$$B_{\eta/4}(x_0) \cap \mathrm{SP}(\phi^{i,j}_{n,m})_y \neq \emptyset \ \text{ for all } y \in X_{m,j}.$$

If $x_0 \in X_i^j$ is a fractional point, then not only do we have

$$B_{\eta/4}(x_0) \cap \mathrm{SP}(\phi_{n,m}^{i,j})_y \neq \emptyset \quad \text{for all } y \in X_{m,j}$$

but also $\{x_0\} \cap \mathrm{SP}(\phi_{n,m}^{i,j})_y \neq \emptyset$, that is, $x_0 \in \mathrm{SP}(\phi_{n,m}^{i,j})_y$ for all $y \in X_{m,j}$. Thus the claim is true.

Now let us enlarge X_i^j. Let $X'^j_i = \{x \in X_i^j : x \text{ is not fractional}\}$ (where 0 is a single point not the set $\{0_1, \cdots 0_{r_0}\}$). Since X'^j_i is still a closed set in the ordinary closed interval $[0,1]$, there exist $\{x_k\}_{k=1}^L \subseteq X'^j_i$ such that

$$X'^j_i \subseteq \bigcup_{k=1}^{L} B_{\eta/4}(x_k) \triangleq \widetilde{X}_i^j.$$

(1) If $0 \in \widetilde{X}_i^j$, $1 \in \widetilde{X}_i^j$, then this $\widetilde{X}_i^j$ is what we expected.
(2) If $0 \notin \widetilde{X}_i^j$, then let

new $\widetilde{X}_i^j = \widetilde{X}_i^j \cup \{x \in X_i^j : x = 0_k \text{ for some } k\}$. We still denote it by $\widetilde{X}_i^j$.

(3) If $1 \notin \widetilde{X}_i^j$, then let

new $\widetilde{X}_i^j = \widetilde{X}_i^j \cup \{x \in X_i^j : x = 1_k \text{ for some } k\}$. Again we still denote it by $\widetilde{X}_i^j$.

Therefore, $\widetilde{X}_i^j$ is a finite disjoint union of intervals or fractional points. Let $Y_i^{j,1}, \cdots, Y_i^{j,\bullet}$ ($j = 1, 2, \cdots, k_m$, A_m has k_m blocks) denote all the path connected components of $\widetilde{X}_i^j$. We claim that

$$Y_i^{1,1}, \cdots, Y_i^{1,\bullet}, Y_i^{2,1}, \cdots, Y_i^{j,s}, \cdots, Y_i^{k_m,\bullet}$$

satisfying the following two properties:

Lemma 1.3.8. *If we let* $B = \bigoplus_{i=1}^{k_n} \bigoplus_{j=1}^{k_m} \bigoplus_{s} A_n^i|_{Y_i^{j,s}}$, *then* $\phi_{n,m}$ *can be written as*

$$\phi_{n,m} : A_n \xrightarrow{\pi} B \xrightarrow{\bigoplus_s \phi_s} A_m$$

where $\pi = \bigoplus_s \pi_s$, $\pi_s(f) = f|_{Y_i^{j,s}}$ *and* $\phi_s : A_m^i|_{Y_i^{j,s}} \to A_m^j$ *is the homomorphism induced by* $\phi_{n,m}^{i,j}$.

Lemma 1.3.9. *We have*

$$\mathrm{SP}(\phi_s)_y \cap B_{\eta/2}(x_0, Y_i^{j,s}) \neq \emptyset, \ \textit{for all } x_0 \in Y_i^{j,s}, \ y \in X_m^j.$$

Lemma 1.3.8 is obvious by the above discussion. Now we prove Lemma 1.3.9.

Proof. Note that if $x_0 \in Y_i^{j,s}$ is a fractional point ($x_0 = 0_k$ for example) and $0 \notin Y_i^{j,s}$, then $x_0 \in \mathrm{SP}(\phi_s)_y$ for all $y \in X_m^j$. Therefore, $\mathrm{SP}(\phi_s)_y \bigcap B_{\eta/2}(x_0, Y_i^{j,s}) \neq \emptyset$.

If $x_0 \in Y_i^{j,s}$ and x_0 is not a fractional point, then by the process of the construction above, there exists $x_k \in X'^j_i \subseteq X_i^j$, such that

$$x_0 \in B_{\eta/4}(x_k) \subseteq Y_i^{j,s}.$$

Notice that

$$\begin{aligned} \mathrm{SP}(\phi_s)_y &= \mathrm{SP}(\phi'^{i,j}_{n,m})_y \cap Y_i^{j,s} \\ &= \mathrm{SP}(\phi^{i,j}_{n,m})_y \cap Y_i^{j,s} \quad \text{for all } y \in X_{m,j}. \end{aligned}$$

In addition, by the claim we have proved above, we know that $B_{\eta/4}(x_k) \subset Y_i^{j,s}$, and

$$\begin{aligned} \mathrm{SP}(\phi_s)_y \cap B_{\eta/2}(x_0, Y_i^{j,s}) &= \mathrm{SP}(\phi^{i,j}_{n,m})_y \cap Y_i^{j,s} \cap B_{\eta/2}(x_0, Y_i^{j,s}) \\ &\supseteq \mathrm{SP}(\phi^{i,j}_{n,m})_y \cap B_{\eta/4}(x_k) \cap Y_i^{j,s} \\ &= \mathrm{SP}(\phi^{i,j}_{n,m})_y \cap B_{\eta/4}(x_k) \neq \emptyset \quad \text{for all } y \in X_{m,j}. \end{aligned}$$

This completes our proof. □

As in the proof of Theorem 4.2 in [30], we have the following theorem which we call the **Dichotomy Theorem**.

Theorem 1.3.10. *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ be a C^*-algebra with the ideal property, where $A_n \subset \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_n^i))$ are splitting interval algebras, $X_n^i = [0,1]$. For any fixed A_n, and any $\eta > 0$, there exist $\delta > 0$, a positive integer $m_0 > n$, subintervals or fractional point sets $Y_i^1, \cdots, Y_i^\bullet \subset X_n^i$, $i = 1, 2, \cdots, k_n$, and a homomorphism*

$$\phi : B = \bigoplus_{i=1}^{k_n} \bigoplus_s A_n^i|_{Y_i^s} \to A_m, \ (m > m_0)$$

such that

(1) $\phi_{n,m}$ factors as

$$\phi_{n,m} : A_n \xrightarrow{\pi} B \xrightarrow{\phi} A_m \ ,$$

where $\pi(f) = (f|_{Y_i^1}, f|_{Y_i^2}, \cdots, f|_{Y_i^\bullet}) \in B$, $\forall f \in A_n^i$;

(2) The homomorphism ϕ satisfies the dichotomy condition, i.e., for all Y_i^s, the partial map $\phi_s = \phi_i^{j,s} : A_n^i|_{Y_i^s} \to A_m^j$ is either zero or has the property $\mathrm{sdp}(\eta, \delta)$. Additionally, for any $m' > m$, each $\phi_{m,m'} \circ \phi$ also satisfies the dichotomy condition.

Remark 1.3.11. From the process of the proof of Lemma 1.3.8, we can see that $A_n^i|_{Y_i^s}$ could be either a splitting interval algebra (of the form $A_n^i|_{[0,\delta]}$ or $A_n^i|_{[1-\delta,1]}$ for some δ), or an interval algebra $A_n^i|_{[\alpha,\beta]}$ with $0 < \alpha < \beta < 1$, or $\mathrm{M}_{n_{x_k}} = A_n^i|_{x_k}$ for $x = 0$ or 1 and some k. In the latter case, there is an $\varepsilon > 0$ such that $(0, \varepsilon) \cap \mathrm{SP}\phi^{i,j}_{n,m} = \emptyset$ or $(1-\varepsilon, 1) \cap \mathrm{SP}\phi^{i,j}_{n,m} = \emptyset$.

1.4 Classification

Theorem 1.4.1. *For C^*-algebras $A = \lim\limits_{n\to\infty}(A_n, \phi_{n,m})$, $B = \lim\limits_{n\to\infty}(B_n, \psi_{n,m})$ with the ideal property where $A_n = \bigoplus\limits_{i=1}^{k_n} A_n^i$ and $B_m = \bigoplus\limits_{j=1}^{l_m} B_m^j$, A_n^i and B_m^j are splitting interval algebras satisfying the following conditions:*

(1) There exists a scaled ordered isomorphism $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$;

(2)Let $\mathcal{P}(A)$, $\mathcal{P}(B)$ denote the set of all projections in A and B respectively. For any $e \in \mathcal{P}(A)$, $f \in \mathcal{P}(B)$ with $\alpha[e] = [f]$, there exists an isomorphism $\xi^{e,f}$: $\mathrm{AffT}(eAe) \to \mathrm{AffT}(fBf)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{AffT}(eAe) & \xrightarrow{\xi^{e,f}} & \mathrm{AffT}(fBf) \\ \uparrow & & \uparrow \\ \mathrm{AffT}(e'Ae') & \xrightarrow{\xi^{e',f'}} & \mathrm{AffT}(f'Bf') \end{array}$$

i.e. $\xi^{e,f}$ and $\xi^{e',f'}$ are compatible, for any $e' \in \mathcal{P}(A)$, $f' \in \mathcal{P}(B)$ satisfying $e' < e$, $f' < f$, and $\alpha[e'] = [f']$.

Then there exists an isomorphism $\Gamma : A \to B$ such that

(1) $\mathrm{K}_0(\Gamma) = \alpha$;

(2) If we let $\Gamma_e : eAe \to \Gamma(e)B\Gamma(e)$ be the restriction of Γ in eAe, then

$$\mathrm{AffT}(\Gamma_e) = \xi^{e,f}, \quad \forall [f] = [\Gamma(e)].$$

Remark 1.4.2. By the condition (2) of Theorem 1.4.1, we can deduce that α and $\xi^{e,f}$ are compatible (see 1.19 in [30]).

Let $A = \lim\limits_{n\to\infty}(A_n, \phi_{n,m})$, $B = \lim\limits_{n\to\infty}(B_n, \psi_{n,m})$ satisfy the assumptions of Theorem 1.4.1. Suppose that $\xi : \mathrm{AffT}A \to \mathrm{AffT}B$, $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ are both scaled ordered isomorphisms such that α and ξ are compatible. By Lemma 1.2.1, there exists an intertwining at the K_0 stage:

$$\begin{array}{ccccccccc} \mathrm{K}_0A_1 & \longrightarrow & \mathrm{K}_0A_2 & \longrightarrow & \mathrm{K}_0A_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\ \downarrow{\scriptstyle \alpha_1} & \nearrow{\scriptstyle \beta_1} & \downarrow{\scriptstyle \alpha_2} & \nearrow{\scriptstyle \beta_2} & \downarrow{\scriptstyle \alpha_3} & & & & \downarrow{\scriptstyle \alpha} \\ \mathrm{K}_0B_1 & \longrightarrow & \mathrm{K}_0B_2 & \longrightarrow & \mathrm{K}_0B_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B \end{array}$$

where α_i, β_i are all scaled ordered group homomorphisms and there exist homomorphisms $\tilde{\Lambda}_i : A_i \to B_i$, $\tilde{\mathcal{M}}_i : B_i \to A_{i+1}$ such that $\mathrm{K}_0(\tilde{\Lambda}_i) = \alpha_i$, $\mathrm{K}_0(\tilde{\mathcal{M}}_i) = \beta_i$. Since our Existence Theorem and Uniqueness Theorem are for unital algebras, we need to construct a new inductive system to make the homomorphism unital.

To establish this, we use projections to cut down each summand of the original inductive sequence. Following is the process for constructing the new inductive system:

For fixed A_n^i, let $1_{A_n^i}$ denote the unit of A_n^i, define
$[A_{n+k}]_i = \phi_{n,n+k}(1_{A_n^i})A_{n+k}\phi_{n,n+k}(1_{A_n^i}),\ k = 1, 2, \cdots,$
$e_i = \phi_{n,\infty}(1_{A_n^i}),\ e_iAe_i = \phi_{n,\infty}(1_{A_n^i})A\phi_{n,\infty}(1_{A_n^i})$, and
$[B_n]_i = \tilde{\Lambda}_i(1_{A_n^i})B_n\tilde{\Lambda}_i(1_{A_n^i}),\ [B_{n+k}]_i = \psi_{n,n+k}(\tilde{\Lambda}_i(1_{A_n^i}))B_{n+k}\psi_{n,n+k}(\tilde{\Lambda}_i(1_{A_n^i})),$
$k = 1, 2, \cdots,\ f_i = \psi_{n,\infty}(\tilde{\Lambda}_i(1_{A_n^i}))$.
Then we can get the new inductive limits

$$e_iAe_i = \lim_{k\to\infty}([A_{n+k}]_i, [\phi_{n+k,n+l}]_i),\ f_iBf_i = \lim_{k\to\infty}([B_{n+k}]_i, [\psi_{n+k,n+l}]_i)$$

where $[\phi_{n+k,n+l}]_i,\ [\psi_{n+k,n+l}]_i$ denote the unital homomorphisms induced by $\phi_{n+k,n+l}$, $\psi_{n+k,n+l}$ respectively. We can also get the intertwining:

$$\begin{array}{ccccccccc}
\mathrm{K}_0[A_n]_i & \longrightarrow & \mathrm{K}_0[A_{n+1}]_i & \longrightarrow & \mathrm{K}_0[A_{n+2}]_i & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0(e_iAe_i) \\
\downarrow{\scriptstyle [\alpha_n]_i} & \nearrow{\scriptstyle [\beta_n]_i} & \downarrow{\scriptstyle [\alpha_{n+1}]_i} & \nearrow{\scriptstyle [\beta_{n+1}]_i} & \downarrow{\scriptstyle [\alpha_{n+2}]_i} & & & & \downarrow{\scriptstyle \alpha^{e_i,f_i}} \\
\mathrm{K}_0[B_n]_i & \longrightarrow & \mathrm{K}_0[B_{n+1}]_i & \longrightarrow & \mathrm{K}_0[B_{n+2}]_i & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0(f_iBf_i)
\end{array}$$

where $[\alpha_{n+k}]_i, [\beta_{n+k}]_i, \alpha^{e_i,f_i}(k = 0, 1, \cdots)$ are all scaled ordered, and $\alpha^{e_i,f_i}[e_i] = [f_i]$.

Similarly, for fixed B_m^j, we can get another set of inductive limits $\tilde{f}_jB\tilde{f}_j,\ \tilde{e}_jA\tilde{e}_j$, where $\tilde{f}_j = \psi_{m,\infty}(1_{B_m^j}),\ \tilde{e}_j = \phi_{m+1,\infty}\tilde{\mathcal{M}}_m(1_{B_m^j})$ and $\alpha[\tilde{e}_j] = [\tilde{f}_j]$. If we let

$$\{B_m\}_j = B_m^j,\ \{B_{m+k}\}_j = \psi_{m,m+k}(1_{B_m^j})B_{m+k}\psi_{m,m+k}(1_{B_m^j}),$$

and $\{A_{m+k}\}_j = \phi_{m+1,m+k}(\tilde{\mathcal{M}}_m(1_{B_m^j}))A_{m+k}\phi_{m+1,m+k}(\tilde{\mathcal{M}}_m(1_{B_m^j})),\ k = 1, 2, \cdots$, then we get the new inductive limits

$$\tilde{e}_jA\tilde{e}_j = \lim_{k\to\infty}(\{A_{m+k}\}_j, \{\phi_{m+k,m+l}\}_j),\ \tilde{f}_jB\tilde{f}_j = \lim_{k\to\infty}(\{B_{m+k}\}_j, \{\psi_{m+k,m+l}\}_j),$$

where $\{\phi_{m+k,m+l}\}_j,\ \{\psi_{m+k,m+l}\}_j$ denote the unital homomorphisms induced by $\phi_{m+k,m+l},\ \psi_{m+k,m+l}$ respectively.

Later we will discuss the cut down algebras $q_sB_m^jq_s$ for a set of mutually orthogonal projections q_s. Considering $q_sB_m^jq_s$ instead of B_m^j, as above we will get inductive limits

$$\tilde{e}_{s,j}A\tilde{e}_{s,j} = \lim_{k\to\infty}(\{A_{m+k}\}_{s,j}, \{\phi_{m+k,m+l}\}_{s,j}),\ \tilde{f}_{s,j}B\tilde{f}_{s,j} = \lim_{k\to\infty}(\{B_{m+k}\}_{s,j}, \{\psi_{m+k,m+l}\}_{s,j}),$$

and $\tilde{e}_{s,j} < \widetilde{e}_j,\ \tilde{f}_{s,j} < \widetilde{f}_j,\ \alpha[\tilde{e}_{s,j}] = [\tilde{f}_{s,j}]$, where the symbols $\tilde{e}_{s,j},\ \tilde{f}_{s,j},\ \{A_{m+k}\}_{s,j}, \{B_{m+k}\}_{s,j}$ can be defined as above.

The symbols $[\ ]_i,\ \{\ \}_j$ always denote the algebras cut down by the image of the unit of $A_n^i,\ B_m^j$, respectively or the related unital homomorphisms.

Using the definitions and symbols mentioned above, we have the following lemmas.

Lemma 1.4.3. *Let* $\{q_s\}_{s=1}^{\bullet}$ *be finite orthogonal non-zero projections in* $B_{m_1}^j$, *and* $F_s \subset \mathrm{AffT}(q_s B_{m_1}^j q_s)$ *be finite sets. For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *and a finite set* $G \subset \mathrm{AffT}B_{m_1}^j$ *such that the following statement is true:*

If a unital homomorphism $\mathcal{M}_j : B_{m_1}^j \to \{A_{n_2}\}_j$ *satisfies that*

$$||\mathrm{AffT}\{\phi_{n_2,\infty}\}_j \circ \mathrm{AffT}\mathcal{M}_j(g) - (\xi^{\tilde{e}_j,\tilde{f}_j})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_j(g)|| < \delta,\ \forall g \in G,$$

then the unital homomorphism $\mathcal{M}_{s,j} : q_s B_{m_1}^j q_s \to \{A_{n_2}\}_{s,j}$ *induced by* $\mathcal{M}_j$ *satisfies*

$$||\mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j} \circ \mathrm{AffT}\mathcal{M}_{s,j}(f) - (\xi^{\tilde{e}_{s,j},\tilde{f}_{s,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f)|| < \varepsilon,\ \forall f \in F_s.$$

Proof. Let $I_s : q_s B_{m_1}^j q_s \to B_{m_1}^j$ be the embedding map, and $G \triangleq \bigcup_s \mathrm{AffT}I_s(F_s)$. Let

$$\delta = \varepsilon \cdot \min_s \min_{\substack{1 \le i \le r_0 \\ 1 \le j \le r_1}} \{\frac{k_{0_i}(s)}{n_{0_i}},\ \frac{k_{1_j}(s)}{n_{1_j}},\ \sum_{i=1}^{r_0} \frac{k_{0_i}(s)}{n} : k_{0_i}(s) \ne 0,\ k_{1_j}(s) \ne 0\},$$

where

$$k_0(q_s) = (\bar{k}_0(s), \bar{k}_1(s)) = (k_{0_1}(s), \cdots, k_{0_{r_0}}(s), k_{1_1}(s), \cdots, k_{1_{r_1}}(s))$$

as in Lemma 1.1.9. Suppose that there exists a unital homomorphism $\mathcal{M}_j$ satisfying

$$||\mathrm{AffT}\{\phi_{n_2,\infty}\}_j \circ \mathrm{AffT}\mathcal{M}_j(g) - (\xi^{\tilde{e}_j,\tilde{f}_j})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_j(g)|| < \delta,\ \forall g \in G.$$

By definition of G, we have $AffI_s(f) \in G$, for all $f \in F_s$, so for all $f \in F_s$,

$$\triangle_s \triangleq ||\mathrm{AffT}\{\phi_{n_2,\infty}\}_j \circ \mathrm{AffT}\mathcal{M}_j(AffI_s(f)) - (\xi^{\tilde{e}_j,\tilde{f}_j})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_j(AffI_s(f))|| < \delta.$$

We already know that the following diagrams commute in a natural way:

$$\begin{array}{ccc} \mathrm{AffT}B_{m_1}^j & \xrightarrow{\mathrm{AffT}\mathcal{M}_j} & \mathrm{AffT}\{A_{n_2}\}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\{B_{m_1}\}_{s,j} & \xrightarrow{\mathrm{AffT}\mathcal{M}_{s,j}} & \mathrm{AffT}\{A_{n_2}\}_{s,j} \end{array} \qquad (4.1),$$

$$\begin{array}{ccc} \mathrm{AffT}B_{m_1}^j & \xrightarrow{\mathrm{AffT}\{\psi_{m_1,\infty}\}_j} & \mathrm{AffT}\tilde{f}_j B \tilde{f}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\{B_{m_1}\}_{s,j} & \xrightarrow{\mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}} & \mathrm{AffT}\tilde{f}_{s,j} B \tilde{f}_{s,j} \end{array} \qquad (4.2),$$

$$\begin{array}{ccc} \mathrm{AffT}\{A_{n_2}\}_j & \xrightarrow{\mathrm{AffT}\{\phi_{n_2,\infty}\}_j} & \mathrm{AffT}\tilde{e}_j A \tilde{e}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\{A_{n_2}\}_{s,j} & \xrightarrow{\mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j}} & \mathrm{AffT}\tilde{e}_{s,j} A \tilde{e}_{s,j} \end{array} \qquad (4.3),$$

$$\begin{array}{ccc} \mathrm{AffT}\tilde{e}_j A \tilde{e}_j & \xrightarrow{\xi^{\tilde{e}_j,\tilde{f}_j}} & \mathrm{AffT}\tilde{f}_j B \tilde{f}_j \\ \uparrow & & \uparrow \\ \mathrm{AffT}\tilde{e}_{s,j} A \tilde{e}_{s,j} & \xrightarrow{\xi^{\tilde{e}_{s,j},\tilde{f}_{s,j}}} & \mathrm{AffT}\tilde{f}_{s,j} B \tilde{f}_{s,j} \end{array} \qquad (4.4),$$

where the last diagram arises from the compatibility of $\mathrm{AffT}eAe$ and $\mathrm{AffT}e'Ae'(e' < e)$ which is the condition (2) in Theorem 1.4.1.

By the commutativity of Diagrams (4.1) and (4.2), we have

$$\triangle_s = ||\mathrm{AffT}\{\phi_{n_2,\infty}\}_j \circ \mathrm{AffT}I_s \circ \mathrm{AffT}\mathcal{M}_{s,j}(f) - (\xi^{\tilde{e}_j,\tilde{f}_j})^{-1} \circ \mathrm{AffT}I_s \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f)|| < \delta,$$

where we denote all the embedding maps by I_s.

By the commutativity of Diagrams (4.3) and (4.4), we have

$$\triangle_s = ||\mathrm{AffT}I_s \circ (\mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j} \circ \mathrm{AffT}\mathcal{M}_{s,j}(f) - (\xi^{\tilde{e}_{s,j},\tilde{f}_{s,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f))|| < \delta.$$

Let $f' = \mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j} \circ \mathrm{AffT}\mathcal{M}_{s,j}(f) - (\xi^{\tilde{e}_{s,j},\tilde{f}_{s,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f)$. By 1.1.13, we have

$$\mathrm{AffT}I_s(f')(0_i) = \frac{k_{0_i}(s)}{n_{0_i}} f'(0_i),\ \mathrm{AffT}I_s(f')(1_j) = \frac{k_{1_j}(s)}{n_{1_j}} f'(1_j),$$

where $k_{0_i}(s) \neq 0,\ k_{1_j}(s) \neq 0$. By 1.1.13, we know

$$\mathrm{AffT}I_s(f')(x) = \frac{\sum_{i=1}^{r_0} k_{0_i}(s)}{n} f'(x),$$

where $x \in [0,1],\ k_0(q_s) = (\bar{k}_0(s), \bar{k}_1(s)) = (k_{0_1}(s), \cdots, k_{0_{r_0}}(s), k_{1_1}(s), \cdots, k_{1_{r_1}}(s))$. This shows that

$$\delta > ||\mathrm{AffT}I_s(f')|| \geq \min_{\substack{1 \leq i \leq r_0 \\ 1 \leq j \leq r_1}} \{\frac{k_{0_i}(s)}{n_{0_i}}, \frac{k_{1_j}(s)}{n_{1_j}}, \sum_{i=1}^{r_0} \frac{k_{0_i}(s)}{n} : k_{0_i}(s) \neq 0,\ k_{1_j}(s) \neq 0\} ||f'||,$$

$$\forall f' \in \mathrm{AffT}\tilde{e}_{s,j} A \tilde{e}_{s,j}.$$

Since

$$\delta = \varepsilon \cdot \min_s \min_{\substack{1 \leq i \leq r_0 \\ 1 \leq j \leq r_1}} \{\frac{k_{0_i}(s)}{n_{0_i}}, \frac{k_{1_j}(s)}{n_{1_j}}, \sum_{i=1}^{r_0} \frac{k_{0_i}(s)}{n} : k_{0_i}(s) \neq 0,\ k_{1_j}(s) \neq 0\},$$

we also have

$$||\mathrm{AffT}\{\phi_{n_2,\infty}\}_{s,j} \circ \mathrm{AffT}\mathcal{M}_{s,j}(f) - (\xi^{\tilde{e}_{s,j},\tilde{f}_{s,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{s,j}(f)|| < \varepsilon,\ \forall f \in F_s.$$

This completes the proof.

□

Lemma 1.4.4. *Let* $A=\lim\limits_{n\to\infty}(A_n,\phi_{n,m})$, $B=\lim\limits_{n\to\infty}(B_n,\psi_{n,m})$ *satisfy the assumptions of Theorem 1.4.1. For fixed* A_{n_1}, *let* $F_i\subset \mathrm{AffT}A^i_{n_1}$ *be finite sets,* $i=1,2,\cdots,k_{n_1}$, *and* $\varepsilon>0$. *Then there exist* m_1 *and homomorphisms* $\Lambda^i_1: A^i_{n_1}\to [B_{m_1}]_i$ *with the following properties:*

(1) $\mathrm{K}_0\Lambda^{\mathrm{i}}_1=\mathrm{K}_0[\psi_{\mathrm{n_1,m_1}}]_{\mathrm{i}}\circ[\alpha_{\mathrm{n_1}}]_{\mathrm{i}}$, *and*

(2) $||\mathit{AffT}[\psi_{m_1,\infty}]_i\circ \mathit{AffT}\Lambda^i_1(f)-(\xi^{e_i,f_i})\circ \mathit{AffT}[\phi_{n_1,\infty}]_i(f)||<\varepsilon/4,\ \forall f\in F_i.$

Proof. We know that inductive limits

$$e_iAe_i=\lim_{k\to\infty}([A_{n_1+k}]_i,[\phi_{n_1+k,n_1+l}]_i),\ \text{and}\ \ f_iBf_i=\lim_{k\to\infty}([B_{n_1+k}]_i,[\psi_{n_1+k,n_1+l}]_i)$$

are unital. For $A^i_{n_1}$ and finite set F_i, applying the existence theorem, there exists an unital homomorphism $\bar\Lambda^i_1: A^i_{n_1}\to [B_{K_i}]_i\triangleq \bar\Lambda^i_1(1_{A^i_{n_1}})B_{K_i}\bar\Lambda^i_1(1_{A^i_{n_1}})$ such that

$$||\mathrm{AffT}[\psi_{K_i,\infty}]_i\circ \mathrm{AffT}\bar\Lambda^i_1(f)-(\xi^{e_i,f_i})\circ \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)||<\frac{\varepsilon}{4},\ \forall f\in F_i.$$

and $\mathrm{K}_0(\bar\Lambda^i_1)=\mathrm{K}_0[\psi_{n_1,K_i}]_i\circ[\alpha_{n_1}]_i$.

Let $m_1=\max\{K_1,K_2,\cdots,K_{k_{n_1}}\}$, $\Lambda^i_1=[\psi_{K_i,m_1}]_i\circ\bar\Lambda^i_1$. Then

$$\begin{aligned}
&||\mathrm{AffT}[\psi_{m_1,\infty}]_i\circ \mathrm{AffT}\Lambda^i_1(f)-(\xi^{e_i,f_i})\circ \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)||\\
=\ &||\mathrm{AffT}[\psi_{m_1,\infty}]_i\circ \mathrm{AffT}[\psi_{K_i,m_1}]_i\circ \mathrm{AffT}\bar\Lambda^i_1(f)-(\xi^{e_i,f_i})\circ \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)||\\
=\ &||\mathrm{AffT}[\psi_{K_i,\infty}]_i\circ \mathrm{AffT}\bar\Lambda^i_1(f)-(\xi^{e_i,f_i})\circ \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)||<\tfrac{\varepsilon}{4},\quad \forall f\in F_i.
\end{aligned}$$

and

$$\mathrm{K}_0\Lambda^i_1=\mathrm{K}_0([\psi_{K_i,m_1}]_i\circ\bar\Lambda^i_1)=\mathrm{K}_0[\psi_{K_i,m_1}]_i\circ\mathrm{K}_0[\psi_{n_1,K_i}]_i\circ[\alpha_{n_1}]_i=\mathrm{K}_0[\psi_{n_1,m_1}]_i\circ[\alpha_{n_1}]_i.$$

□

Remark 1.4.5. By Lemma 1.4.4, we may define $\Lambda_1=\bigoplus\limits_i\Lambda^i_1: A_{n_1}=\bigoplus\limits_i A^i_{n_1}\to B_{m_1}$.

Similar to the proof of Lemma 1.4.4, we can prove the following lemma.

Lemma 1.4.6. *Let* $A=\lim\limits_{n\to\infty}(A_n,\phi_{n,m})$, $B=\lim\limits_{n\to\infty}(B_n,\psi_{n,m})$ *be described as in Theorem 1.4.1. Then for any fixed* B_{m_1}, *and finite sets* $G_j\subset\mathrm{AffT}B^j_{m_1}$, $j=1,2,\cdots,l_{m_1}$ *and* $\delta>0$, *there exist* n'_2 *and homomorphisms* $\mathcal{M}^j_1: B^j_{m_1}\to\{A_{n'_2}\}_j$ *satisfying:*

(1) $\mathrm{K}_0\mathcal{M}^j_1=\mathrm{K}_0\{\phi_{m_1+1,n'_2}\}_j\circ\{\beta_{m_1}\}_j$;

(2) $||\mathrm{AffT}\{\phi_{n'_2,\infty}\}_j\circ\mathrm{AffT}\mathcal{M}^j_1(g)-(\xi^{\tilde e_j,\tilde f_j})^{-1}\circ\mathrm{AffT}\{\psi_{m_1,\infty}\}_j(g)||<\delta,\ \forall g\in G_j.$

Lemma 1.4.7. *Let* $A=\lim\limits_{n\to\infty}(A_n,\phi_{n,m})$, $B=\lim\limits_{n\to\infty}(B_n,\psi_{n,m})$ *be described as in Theorem 1.4.1. For fixed* A_{n_1}, *finite sets* $F_i\subset\mathrm{AffT}A^i_{n_1}$, $\varepsilon>0$ *and the homomorphisms* $\Lambda^i_1: A^i_{n_1}\to[B_{m_1}]_i$, $i=1,2,\cdots,k_{n_1}$ *described as in Lemma 1.4.4, then there exist finite sets* $G_j\subset\mathrm{AffT}B^j_{m_1}$, $j=1,2,\cdots,l_{m_1}$, $\delta>0$ *such that:*

If the homomorphisms $\mathcal{M}_1^j : B_{m_1}^j \to \{A_{n_2'}\}_j$ *satisfy the properties in 1.4.6 for* G_j *and* δ*, then there exists* $n_2 > 0$ *such that the homomorphism* $\mathcal{M}_1 \triangleq [\phi_{n_2',n_2}]_i \circ \bigoplus_j \mathcal{M}_1^j : [B_{m_1}]_i \to [A_{n_2}]_i$ *satisfies the following conditions:*

(1) $\mathrm{K}_0[\mathcal{M}_1 \circ \Lambda_1]_i = \mathrm{K}_0[\phi_{n_1,n_2}]_i$*;*

(2) $||\mathrm{AffT}[\phi_{n_1,n_2}]_i(f) - \mathrm{AffT}[\mathcal{M}_1 \circ \Lambda_1]_i(f)|| < \varepsilon,\ \forall f \in F_i,$
where $\Lambda_1 = \bigoplus_i \Lambda_1^i$ *and* $[\mathcal{M}_1 \circ \Lambda_1]_i : A_{n_1}^i \to [\mathcal{M}_1 \circ \Lambda_1]_i(1_{A_{n_1}^i})A_{n_2}[\mathcal{M}_1 \circ \Lambda_1]_i(1_{A_{n_1}^i})$ *are unital homomorphisms.*

Proof. Let Λ_1^i and Λ_1 be as in Lemma 1.4.4 and $\Lambda_1^{i,j} : A_{n_1}^i \to \Lambda_1^{i,j}(1_{A_{n_1}^i})B_{m_1}^j\Lambda_1^{i,j}(1_{A_{n_1}^i})$ be partial maps of Λ_1^i.

Let $G_{i,j} \triangleq \mathrm{AffT}I_{i,j}(\mathrm{AffT}\Lambda_1^{i,j}(F_i))$, $G_j = \bigcup_i G_{i,j} \subseteq \mathrm{AffT}B_{m_1}^j$, where $I_{i,j}$ is the imbedding map from $\Lambda_1^{i,j}(1_{A_{n_1}^i})B_{m_1}^j\Lambda_1^{i,j}(1_{A_{n_1}^i})$ to $B_{m_1}^j$.

For $\delta > 0$ and the finite set $G_j \subseteq \mathrm{AffT}B_{m_1}^j$, by the statement of 1.4.6, we can find a unital homomorphism $\mathcal{M}_1^j : B_{m_1}^j \to \{A_{n_2'}\}_j$ such that

$$||\mathrm{AffT}\{\phi_{n_2',\infty}\}_j \circ \mathrm{AffT}\mathcal{M}_1^j(g) - (\xi^{\tilde{e}_j,\tilde{f}_j})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_j(g)|| < \delta,\ \forall g \in G_j,$$

where $\delta = \frac{\varepsilon}{4}\min_i \min_{s,t}\left\{\frac{k_{0_s}(q_{i,j})}{n_{0_s}(q_{i,j})},\ \frac{k_{1_t}(q_{i,j})}{n_{1_t}(q_{i,j})},\ \sum_{s=1}^{r_0}\frac{k_{0_s}(q_{i,j})}{n(q_{i,j})} : k_{0_s}(q_{i,j}) \neq 0,\ k_{1_t}(q_{i,j}) \neq 0\right\}$ as chosen in Lemma 1.4.3 for any given $\frac{\varepsilon}{4}$. The symbols $q_{i,j} = \Lambda_1^{i,j}(1_{A_{n_1}^i})$ represent projections; $n_{0_s}(q_{i,j})$, $n_{1_t}(q_{i,j})$ and $n(q_{i,j})$ are the sizes of the corresponding matrices of the algebra $B_{m_1}^j$ at the fractional points 0_s, 1_t and full points in $[0,1]$ respectively.

Let

$$\mathcal{M}_1^{i,j} : \Lambda_1^{i,j}(1_{A_{n_1}^i})B_{m_1}^j\Lambda_1^{i,j}(1_{A_{n_1}^i}) \to \mathcal{M}_1^{i,j} \circ \Lambda_1^{i,j}(1_{A_{n_1}^i})A_{n_2'}\mathcal{M}_{i,j} \circ \Lambda_1^{i,j}(1_{A_{n_1}^i})$$

be the unital homomorphism induced by $\mathcal{M}_1^j$. With projections $\{\Lambda_1^{i,j}(1_{A_{n_1}^i})\}_{i=1}^{\bullet} = \{q_s\}_{s=1}^{\bullet}$ (here let $i = s$), since δ and finite set G_j defined above are exactly as chosen in the proof of Lemma 1.4.3, we can use Lemma 1.4.3 to get

$$||\mathrm{AffT}\{\phi_{n_2',\infty}\}_{i,j} \circ \mathrm{AffT}\mathcal{M}_1^{i,j}(g) - (\xi^{\tilde{e}_{i,j},\tilde{f}_{i,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{i,j}(g)|| < \frac{\varepsilon}{4},\ \forall g \in \mathrm{AffT}\Lambda_1^{i,j}(F_i).$$

Let $\mathcal{M}'^i$ be the restriction of $\bar{\mathcal{M}}_1 \triangleq \bigoplus_j \mathcal{M}_1^j$ on $\Lambda_1^i(1_{A_{n_1}^i})B_{m_1}\Lambda_1^i(1_{A_{n_1}^i})$. Then $\mathcal{M}'^i =$

$\bigoplus\limits_j \mathcal{M}_1^{i,j}$, and for all $f \in F_i$, we have

$$\begin{aligned}
& ||\mathrm{AffT}([\phi_{n_2',\infty}]_i \circ \mathcal{M}'^i \circ \Lambda_1^i)(f) - (\xi^{e_i,f_i})^{-1} \circ \mathrm{AffT}([\psi_{m_1,\infty}]_i \circ \Lambda_1^i(f))|| \\
= & ||\mathrm{AffT}([\phi_{n_2',\infty}]_i \circ \mathcal{M}'^i)(\bigoplus_j \mathrm{AffT}\Lambda_1^{i,j}(f)) - (\xi^{e_i,f_i})^{-1} \circ \mathrm{AffT}([\psi_{m_1,\infty}]_i)(\bigoplus_j \mathrm{AffT}\Lambda_1^{i,j}(f))|| \\
\leq & \max_j ||\mathrm{AffT}[\phi_{n_2',\infty}]_i \circ \mathrm{AffT}\mathcal{M}'^i(\mathrm{AffT}\bar{I}_{i,j}(\mathrm{AffT}\Lambda_1^{i,j}(f))) \\
& -(\xi^{e_i,f_i})^{-1} \circ \mathrm{AffT}([\psi_{m_1,\infty}]_i(\mathrm{AffT}\bar{I}_{i,j}(\mathrm{AffT}\Lambda_1^{i,j}(f)))|| \\
= & \max_j ||\mathrm{AffT}I_{i,j}(\mathrm{AffT}\{\phi_{n_2',\infty}\}_{i,j} \circ \mathrm{AffT}\mathcal{M}_1^{i,j}(\mathrm{AffT}\Lambda_1^{i,j}(f)) \\
& -(\xi^{\tilde{e}_{i,j},\tilde{f}_{i,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{i,j}(\mathrm{AffT}\Lambda_1^{i,j}(f)))|| \\
\leq & \max_j ||\mathrm{AffT}\{\phi_{n_2',\infty}\}_{i,j} \circ \mathrm{AffT}\mathcal{M}_1^{i,j}(g) - (\xi^{\tilde{e}_{i,j},\tilde{f}_{i,j}})^{-1} \circ \mathrm{AffT}\{\psi_{m_1,\infty}\}_{i,j}(g)|| \\
< & \tfrac{\varepsilon}{4},
\end{aligned}$$

where $\widetilde{e}_j = \phi_{m_1+1,\infty}(\widetilde{\mathcal{M}}_{m_1}(1_{B_{m_1}}))$, $\widetilde{e}_{i,j} = \phi_{m_1+1,\infty}(\widetilde{\mathcal{M}}_{m_1}(\Lambda_1^{i,j}(1_{A_{n_1}^i})))$, and $\bar{I}_{i,j} : \{B_{m_1}\}_{j,i} \to [B_{m_1}]_i$ and $I_{i,j} : \widetilde{e}_{i,j}A\widetilde{e}_{i,j} \to \widetilde{e}_jA\widetilde{e}_j$ are the embedding maps.

Then

$$||\xi^{e_i,f_i} \circ \mathrm{AffT}([\phi_{n_2',\infty}]_i \circ \mathcal{M}'^i \circ \Lambda_1^i)(f) - \mathrm{AffT}([\psi_{m_1,\infty}]_i \circ \Lambda_1^i)(f)|| < \frac{\varepsilon}{4}, \quad \forall f \in F_i.$$

By Lemma 1.4.4, we know that for each i,

$$||\mathrm{AffT}[\psi_{m_1,\infty}]_i \circ \mathrm{AffT}\Lambda_1^i(f) - \xi^{e_i,f_i} \circ \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)|| < \frac{\varepsilon}{4}, \quad \forall f \in F_i.$$

Then we can get

$$||\mathrm{AffT}[\phi_{n_2',\infty}]_i \circ \mathrm{AffT}(\mathcal{M}'^i \circ \Lambda_1^i)(f) - \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)|| < \varepsilon/2, \quad \forall f \in F_i.$$

Since $\mathcal{M}_i' \circ \Lambda_1^i = \bar{\mathcal{M}}_1 \circ \Lambda_1^i : A_{n_1}^i \to \bar{\mathcal{M}}_1 \circ \Lambda_1^i(1_{A_{n_1}^i})A_{n_2'}\bar{\mathcal{M}}_1 \circ \Lambda_1^i(1_{A_{n_1}^i})$, then

$$\mathrm{AffT}(\mathcal{M}'^i \circ \Lambda_1^i)(f) = \mathrm{AffT}(\bar{\mathcal{M}}_1 \circ \Lambda_1^i)(f).$$

This implies that

$$||\mathrm{AffT}[\phi_{n_2',\infty}]_i \circ \mathrm{AffT}(\bar{\mathcal{M}}_1 \circ \Lambda_1^i)(f) - \mathrm{AffT}[\phi_{n_1,\infty}]_i(f)|| < \frac{\varepsilon}{2}, \quad \forall f \in F_i.$$

By the definition of inductive limits, there exists $n_2 > 0$ such that:

$$||\mathrm{AffT}[\phi_{n_2',n_2}]_i \circ \mathrm{AffT}(\bar{\mathcal{M}}_1 \circ \Lambda_1^i)(f) - \mathrm{AffT}[\phi_{n_1,n_2}]_i(f)|| < \varepsilon, \quad \forall f \in F_i.$$

As a result, we only need to define $\mathcal{M}_1 = [\phi_{n_2',n_2}]_i \circ \bar{\mathcal{M}}_1$. Using Lemma 1.4.4 and statement of 1.4.6, we have

$$\mathrm{K}_0(\mathcal{M}_1 \circ \Lambda_1^i) = \mathrm{K}_0[\phi_{n_1,n_2}]_i.$$

The proof is completed.

$\square$

Now we prove the Theorem 1.4.1.

Proof. Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$, $B = \lim_{n\to\infty}(B_n, \psi_{n,m})$ be two given C*-algebras with the ideal property, and an isomorphism $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$. There exist scaled ordered group maps $\alpha_i : \mathrm{K}_0 A_i \to \mathrm{K}_0 B_i$, $\beta_i : \mathrm{K}_0 B_i \to \mathrm{K}_0 A_{i+1}$ making the following diagram commutative:

$$\begin{array}{ccccccccc}
\mathrm{K}_0A_1 & \longrightarrow & \mathrm{K}_0A_2 & \longrightarrow & \mathrm{K}_0A_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0A \\
\downarrow{\scriptstyle\alpha_1} & \nearrow{\scriptstyle\beta_1} & \downarrow{\scriptstyle\alpha_2} & \nearrow{\scriptstyle\beta_2} & \downarrow{\scriptstyle\alpha_3} & & & & \downarrow{\scriptstyle\alpha} \\
\mathrm{K}_0B_1 & \longrightarrow & \mathrm{K}_0B_2 & \longrightarrow & \mathrm{K}_0B_3 & \longrightarrow & \cdots & \longrightarrow & \mathrm{K}_0B,
\end{array}$$

and there exist homomorphisms $\tilde{\Lambda}_i : A_i \to B_i$, $\tilde{\mathcal{M}}_i : B_i \to A_{i+1}$ such that $\mathrm{K}_0(\tilde{\Lambda}_i) = \alpha_i$, $\mathrm{K}_0(\tilde{\mathcal{M}}_i) = \beta_i$ (See Lemma 1.2.1).

To prove the classification theorem, we need to construct an approximate intertwining of the two sequences of C*-algebras.

In this process, we will pass to subsequence several times. Let $\varepsilon_1, \varepsilon_2, \cdots$ be positive numbers with $\sum_{i=1}^{\infty} \varepsilon_i < \infty$. We choose the subsequence of $\{A_n\}_{n=1}^{\infty}$, $\{B_m\}_{m=1}^{\infty}$,

$$A_{k_1} \longrightarrow A_{k_2} \longrightarrow \cdots \longrightarrow A,$$

$$B_{l_1} \longrightarrow B_{l_2} \longrightarrow \cdots \longrightarrow B,$$

and maps $\Lambda_i : A_{k_i} \longrightarrow B_{l_i}$, $\mathcal{M}_i : B_{l_i} \longrightarrow A_{k_{i+1}}$ satisfying certain conditions so that the diagram

$$\begin{array}{ccccccccc}
A_{k_1} & \longrightarrow & A_{k_2} & \longrightarrow & A_{k_3} & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow{\scriptstyle\Lambda_1} & \nearrow{\scriptstyle\mathcal{M}_1} & \downarrow{\scriptstyle\Lambda_2} & \nearrow{\scriptstyle\mathcal{M}_2} & \downarrow{\scriptstyle\Lambda_3} & & & & \downarrow \\
B_{l_1} & \longrightarrow & B_{l_2} & \longrightarrow & B_{l_3} & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

is an approximate intertwining, i.e. homomorphisms Λ_i, $\mathcal{M}_i$ and the finite generating subsets $F_{k_i} \subset A_{k_i}$, $G_{l_i} \subset B_{l_i}$ satisfy

$$||\Lambda_i \circ \mathcal{M}_{i-1}(f) - \psi_{l_{i-1},l_i}(f)|| < \varepsilon_i, \ \forall f \in G_{l_{i-1}},$$

$$||\mathcal{M}_i \circ \Lambda_i(f) - \phi_{k_i,k_{i+1}}(f)|| < \varepsilon_i, \ \forall f \in F_{k_i},$$

and

$$F_{k_i} \supseteq \mathcal{M}_{i-1}(G_{l_{i-1}}) \cup \phi_{k_{i-1},k_i}(F_{k_{i-1}}),$$

$$G_{l_i} \supseteq \Lambda_i(F_{k_i}) \cup \psi_{l_{i-1},l_i}(G_{l_{i-1}}).$$

Then by Theorem 2.1 of [13], it follows that A, B are isomorphic.

Now let $F_i \subset A_i$, $G_i \subset B_i$ be finite sets such that

$$F_1 \subset F_2 \subset \cdots \subset \overline{\bigcup_{i=1}^{\infty} F_i} = A,$$

$$G_1 \subset G_2 \subset \cdots \subset \overline{\bigcup_{i=1}^{\infty} G_i} = B.$$

Choose $k_1 = 1$. For $F_1 \subset A_1$, $\varepsilon_1 > 0$, there exists $\eta_1 > 0$ satisfying the Uniqueness Theorem (Theorem 1.3.7). For this η_1, by the Dichotomy Theorem (Theorem 1.3.10), there exist $\delta_1 > 0$ and a positive integer n_1 such that $\phi_{1,n_1} : A_1 \longrightarrow A_{n_1}$ factors as

$$\phi_{1,n_1} : A_1 \xrightarrow{\pi} \left(B = \bigoplus_i \bigoplus_j A_1^i|_{Y_j^s} \right) \xrightarrow{\phi = \bigoplus_s \phi_s} \left(A_{n_1} = \bigoplus_{i'} A_{n_1}^{i'} \right),$$

where ϕ_s has the property $\mathrm{sdp}(\eta_1, \delta_1)$ and each partial map of $\phi_{n_1,m} \circ \phi$ also has the property $\mathrm{sdp}(\eta_1, \delta_1)$, for any $m > n_1$. By the Uniqueness Theorem, there exists a finite set $H(\eta_1, \delta_1, X_1) \subseteq \mathrm{AffT}A_1$ (where $X_1 = [0,1]$).

For each fixed $A_{n_1}^{i'}$, we can find C*-algebras $e_{i'}Ae_{i'}$, $f_{i'}Bf_{i'}$ where $e_{i'} = \phi_{n_1,\infty}(1_{A_{n_1}^{i'}})$, $f_{i'} = \psi_{n_1,\infty}(\widetilde{\Lambda}_{n_1}(1_{A_{n_1}^{i'}}))$, and compatible pair $(\alpha^{e_{i'},f_{i'}}, \xi^{e_{i'},f_{i'}})$ between them. Naturally, $e_{i'}Ae_{i'}$, $f_{i'}Bf_{i'}$ still satisfy the conditions of the existence theorem. Notice that

$$\phi_s = \phi_i^{i',s} : A_1|_{Y_i^{i',s}} \longrightarrow A_{n_1}^{i'},$$

and set

$$E_{i'}^s \triangleq \mathrm{AffT}(\phi_s \circ \pi_s)(H(\eta_1, \delta_1, X_1)), \ E_{i'} = \bigoplus_s E_{i'}^s \subseteq \mathrm{AffT}A_{n_1}^{i'}.$$

For these finite sets $E_{i'}$, applying Lemmas 1.4.4, 1.4.6 and 1.4.7, we can find homomorphisms

$$\Lambda_1^{i'} : A_{n_1}^{i'} \longrightarrow [B_{m_1}]_{i'}, \ \mathcal{M}_1 : [B_{m_1}]_{i'} \longrightarrow [A_{n_2}]_{i'}$$

such that

$$||\mathrm{AffT}[\phi_{n_1,n_2}]_{i'}(f) - \mathrm{AffT}[\mathcal{M}_1 \circ \Lambda_1]_{i'}(f)|| < \delta_1, \ \forall f \in E_{i'},$$

where $\Lambda_1 \triangleq \bigoplus_{i'} \Lambda_1^{i'}$ and $[\mathcal{M}_1 \circ \Lambda_1]_{i'} : A_{n_1}^{i'} \longrightarrow \mathcal{M}_1 \circ \Lambda_1(1_{A_{n_1}^{i'}}) A_{n_2} \mathcal{M}_1 \circ \Lambda_1(1_{A_{n_1}^{i'}})$ are unital. By the definition of $E_{i'}$, for any $f \in \pi_s(H(\eta, \delta, X))$ we have

$$||\mathrm{AffT}([\phi_{n_1,n_2}]_{i'} \circ \phi_s)(f) - \mathrm{AffT}[\mathcal{M}_1 \circ \Lambda_1]_{i'} \circ \mathrm{AffT}\phi_s(f)|| < \delta_1.$$

Since ϕ_s has the property $\mathrm{sdp}(\eta_1, \delta_1)$, partial maps $[\phi_{n_1}, \phi_{n_2}]_i \circ \phi_s$ also have the property $\mathrm{sdp}(\eta_1, \delta_1)$. By the Uniqueness Theorem (Theorem 1.3.7), we can find unitary elements $U_s \in A_{n_2}$ such that

$$||[\phi_{n_1,n_2}]_{i'} \circ \phi_s(f) - U_s([\mathcal{M}_1 \circ \Lambda_1]_{i'} \circ \phi_s)(f) U_s^*|| < \varepsilon_1, \ \forall f \in \pi_s(F_1).$$

Notice that

$$\phi_s = \phi_i^{i',s} : A_1|_{Y_i^{i',s}} \longrightarrow A_{n_1}^{i'} \text{ and } \phi_{1,n_1} = \phi \circ \pi = \bigoplus_s (\phi_s \circ \pi_s).$$

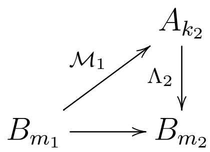
Setting $\bar{\Lambda}_1 = (\bigoplus_{i'} \Lambda_1^{i'}) \circ \phi_{1,n_1}$, $\bar{\mathcal{M}}_1 = (\bigoplus_s U_s)\mathcal{M}_1(\bigoplus_s U_s^*)$, then for each $f \in F_1$, we have

$$\begin{aligned} & ||\phi_{1,n_2}(f) - \bar{\mathcal{M}}_1 \circ \bar{\Lambda}_1(f)|| \\ \leq \ & \max_s ||[\phi_{n_1,\phi_2}]_{i'} \circ \phi_s \circ \pi_s(f) - U_s(\mathcal{M}_1 \circ \Lambda_{i'} \circ \phi_s(f))U_s^*|| < \varepsilon_1. \end{aligned}$$

We still denote $\bar{\Lambda}_1$, $\bar{\mathcal{M}}_1$ by Λ_1 and $\mathcal{M}_1$, so

$$||\phi_{1,n_2}(f) - \mathcal{M}_1 \circ \Lambda_1(f)|| < \varepsilon_1, \quad \forall f \in F_1.$$

Furthermore, when we construct $\mathcal{M}_1$, we can do the same thing as we did in constructing Λ_1, that is, firstly apply Theorem 1.3.7 to get η_2 for G_1 and ε_2; secondly apply the Dichotomy Theorem for η_2 to get δ_2 and the sdp property; and finally apply Lemma 1.4.6 for $\delta' = \min\{\delta_2, \delta\}$, where δ is the δ in Lemma 1.4.3 for $\frac{\delta_1}{4}$. In this way, we can construct the next step of the almost commutative diagram:

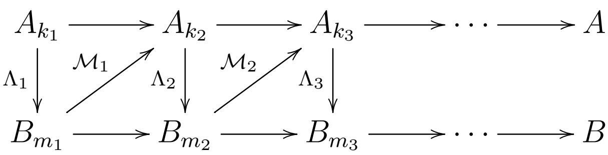


Inductively, we can construct Λ_i, $\mathcal{M}_i$ such that

$$||\Lambda_{i+1} \circ \mathcal{M}_i(f) - \psi_{m_i,m_{i+1}}(f)|| < \varepsilon_i, \ \forall f \in \tilde{G}_{m_i},$$

$$||\mathcal{M}_i \circ \Lambda_i(f) - \phi_{n_i,n_{i+1}}(f)|| < \varepsilon_i, \ \forall f \in \tilde{F}_{n_i},$$

where $\tilde{G}_{m_i} = G_{m_i} \cup \Lambda_i(\tilde{F}_i) \cup \psi_{m_{i-1},m_i}(\tilde{G}_{m_{i-1}})$, $\tilde{F}_{n_i} = F_{n_i} \cup \mathcal{M}_i(\tilde{G}_{m_i}) \cup \phi_{n_{i-1},n_i}(\tilde{F}_{i-1})$. Then



is an approximate intertwining. Hence A, B are isomorphic, and the conclusions (1) and (2) also hold by the proof above. This completes the proof.

□

Chapter 2

Equivalent Invariants

Let A, B be two C*-algebras with the ideal property. In this chapter, we show that if A and B have isomorphic Elliott Invariant, then they have isomorphic Stevens' Invariant and vise versa. Moreover, for $\mathcal{Z}$-absorbing C*-algebra, we give a characterization of Cuntz comparability by lower semi-continuous dimension functions.

2.1 Preliminaries

For convenience of the reader, we recall some definitions and lemmas (see [45] for more details).

Definition 2.1.1. *Let A be a C*-algebra. A weight on A is a function $\phi : A_+ \to [0, +\infty]$ such that*
(i) $\phi(\alpha x) = \alpha\phi(x)$, if $x \in A_+$ and $\alpha \in \mathbb{R}_+$;
(ii) $\phi(x + y) = \phi(x) + \phi(y)$, if x and y belong to A_+.
ϕ is lower semi-continuous if for each $\alpha \in \mathbb{R}_+$ the set $\{x \in A_+ | \phi(x) \le \alpha\}$ is closed.

Definition 2.1.2. *Let A be a C*-algebra. A trace on A is a weight ϕ such that $\phi(u^*xu) = \phi(x)$ for all $x \in A_+$ and all unitary $u \in \tilde{A}$, where $\widetilde{A}$ is the unitization of A.*

Consider the set $A_+^\phi := \{x \in A_+ | \phi(x) < \infty\}$.

Proposition 2.1.3. *(see 5.1.2 of [45]) For each weight ϕ on a C*-algebra A the linear span A^ϕ of A_+^ϕ is a hereditary *-subalgebra of A with $(A^\phi)_+ = A_+^\phi$, and there is a unique extension of ϕ to a positive linear functional on A^ϕ. Moreover, the set $A_2^\phi = \{x \in A | x^*x \in A_+^\phi\}$ is a left ideal of A such that $y^*x \in A^\phi$ for any $x, y \in A_2^\phi$. Finally, for any x, y in A_2^ϕ we have $|\phi(y^*x)|^2 \le \phi(y^*y)\phi(x^*x)$.*

Lemma 2.1.4. *Let ϕ be a trace on a C*-algebra A, then A^ϕ (as defined above) is an ideal of A. Moreover, A_2^ϕ is also an ideal of A.*

Proof. Let $x \in A_+^\phi$, then for each unitary $u \in \tilde{A}$, by the polarization identity

$$u^*x = (x^{1/2}u)^*x^{1/2} = \frac{1}{4}\sum_{k=0}^{3} i^k(1+i^ku)^*x(1+i^ku)$$

and $(1+i^ku)^*x(1+i^ku) \leq 2(x+u^*xu)$ for each k. By assumption, $u^*xu \in A_+^\phi$, and thus $u^*x \in A^\phi$. Since each element in $\tilde{A}$ is the linear combination of unitaries it follows that A^ϕ is an ideal of A. This implies that also A_2^ϕ is an ideal. □

Proposition 2.1.5. *(see 5.2.2 of [45]) If ϕ is a trace on a C^*-algebra A then $\phi(yx) = \phi(xy)$ for each x in A^ϕ and y in $\tilde{A}$. If moreover ϕ is lower semi-continuous then $\phi(x^*x) = \phi(xx^*)$ for all x in A and $\phi(xy) = \phi(yx)$ for all x and y in A_2^ϕ.*

Definition 2.1.6. *We define an equivalence relation in A_+ by setting $x \approx y$ if there is a finite set $\{z_n\}$ in A such that $x = \sum z_n^*z_n$ and $y = \sum z_nz_n^*$. And we use the notation $y \preccurlyeq x$ to mean $y \approx x_1,\ x_1 \leq x$.*

Theorem 2.1.7. *(see 5.2.7 of [45]) Let B be a hereditary C^*-subalgebra of a C^*-algebra A, and let ϕ be a lower semi-continuous weight on B. For each x in A_+ define*

$$\tilde{\phi}(x) = \sup\{\phi(y) | y \in B_+, y \preccurlyeq x\}.$$

Then $\tilde{\phi}$ is a lower semi-continuous trace on A and $\tilde{\phi}|_{B_+}$ is the smallest trace dominating ϕ.

Let us denote by $\mathrm{T}(A)$ the collection of all lower semicontinuous traces of C^*-algebra A. This set is a non-cancellative cone endowed with the operations of pointwise addition and pointwise scalar multiplication by strictly positive real numbers (see [21] for details). Let $\mathrm{T_F}(A)$ denote the set of all finite traces on A.

2.2 Equivalent Invariants

Lemma 2.2.1. *Let $a, b \in A^+$ be such that $||a-b|| < \varepsilon$. Then $(a-\varepsilon)_+ \preccurlyeq b$.*

Proof. By ([33], Lemma 2.2), there is $d \in A$ with $||d|| \leq 1$ and $(a-\varepsilon)_+ = dbd^*$. Hence, $(a-\varepsilon)_+ \sim b^{1/2}d^*db^{1/2} \leq b$. □

Lemma 2.2.2. *Let A be a C*-algebra and $\tau \in TA$ be a lower semi-continuous trace on A. Define an ideal $I_{fin\tau}$ of A by*

$$I_{fin\tau} := \{x \in A : \ \tau(x) < \infty\}.$$

If $p \in \bar{I}_{fin\tau}$ (the closure of $I_{fin\tau}$) is a projection, then $\tau(p) < \infty$.

Proof. If $p \in I_{fin\phi}$, then $\tau(p) < \infty$ by definition. If $p \in \bar{I}_{fin\phi} \setminus I_{fin\phi}$, then there exists an increasing sequence $\{x_i\}_{i=1}^{\infty} \subseteq I_{fin\tau}$ with $\lim_{i\to\infty} x_i = p$. For any $\varepsilon > 0$, there exists x_k such that:

$$||x_k - p|| < \varepsilon.$$

By Lemma 2.2.1 $(p-\varepsilon)_+ \precsim x_k$, which implies

$$\tau((p-\varepsilon)_+) \leq \tau(x_k).$$

Since $(p-\varepsilon)_+ = (1-\varepsilon)p$, $(1-\varepsilon)\tau(p) \leq \tau(x_k) < \infty$, which means $\tau(p) < \infty$. □

Definition 2.2.3. *Let A, B be two C^*-algebras. Let $\alpha : K_0A \to K_0B$ be a scaled ordered homomorphism, and $\xi : TB \to TA$ be an affine map. We say that α and ξ are compatible if*

$$\tau(\alpha(x)) = (\xi(\tau))(x)$$

for all $x \in K_0A$ and $\tau \in TB$.

Lemma 2.2.4. *Let A, B be two C^*-algebras with the ideal property such that* (1) *There exists a scaled ordered isomorphism $\alpha : K_0(A) \to K_0(B)$.* (2) *There is an isomorphism $\xi : TB \to TA$ which is compatible with α. Let $\tau \in T(B)$, and I be the closed ideal of A generated by the set $\{p \in \mathcal{P}(A) : \alpha(p)$ is a projection in $I_{fin\tau}\}$. Then $\phi(x) = +\infty$ for all $x \in A\backslash I$, where $\phi = \xi(\tau)$.*

Proof. Since $I_{fin\phi} = \{x \in A | \phi(x) < +\infty\}$ is an ideal of A and A has the ideal property, $\bar{I}_{fin\phi}$ is generated by projections inside it. For any projection $p \in \bar{I}_{fin\phi}$, by Lemma 2.2.2 $\phi(p) < +\infty$. Since ξ is compatible with α,

$$\tau(\alpha(p)) = \xi(\tau)(p) = \phi(p) < \infty.$$

Therefore, we have shown that if $p \in \bar{I}_{fin\phi}$ then $\alpha(p) \in I_{fin\tau}$, which implies $p \in I$ (by the definition of I). Therefore $\bar{I}_{fin\phi} \subseteq I$. Thus, $A \setminus I \subseteq A \setminus \bar{I}_{fin\phi}$. Hence $\phi(x) = \infty$ if $x \in A \setminus I$. □

Lemma 2.2.5. *Let ϕ and ξ be two lower semi-continuous traces on a C^*-algebra A and e be a projection in A. Suppose that $\phi(x) = \xi(x) < \infty$ for any $x \in (\overline{eAe})_+$, then $\phi(x) = \xi(x)$ for any $x \in \overline{AeA}$, where $\overline{AeA}$ stands for the closed two-sided ideal generated by e.*

Proof. For $x \in A$, let $\psi(x) = \sup\{\phi(y) : y \precsim x, y \in (\overline{eAe})_+\}$. By Theorem 2.1.7, ψ is a lower semi-continuous trace on A and $\phi(x) = \psi(x)$ for all $x \in \overline{eAe}$. Now we want to show that $\phi(x) = \psi(x)$ for all $x \in \overline{AeA}$.

For any $x \in \overline{AeA}, y \in (\overline{eAe})_+$ with $y \precsim x$, we have $\psi(y) = \phi(y) \leq \phi(x)$. Take supremum on both sides we have $\psi(x) \leq \phi(x)$. $(*)$

Let $\Omega = \{\sum_{k=1}^{n} a_k e b_k : a_k,\ b_k \in A,\ n \in \mathbb{Z}\}$ be a subset of $\overline{AeA}$.

Claim 1: $\phi(x) = \psi(x)$ for all $x \in \Omega$.

In fact, for $x = aeb \in A_+$ with $a, b \in A$, $\phi((ae)^*(ae)) = \phi(ea^*ae) < \infty$ and $\phi((eb)^*(eb)) = \phi((eb)(eb)^*) = \phi(ebb^*e) < \infty$. Similarly for ψ. Therefore, by Proposition 2.1.5,

$$\phi(x) = \phi(aeeb) = \phi(ebae) = \psi(ebae) = \psi(aeb) = \psi(x).$$

Thus, $\phi(x) = \psi(x)$ for all $x \in \Omega$.

Claim 2: $\phi(x) = \psi(x)$ for all $x \in \overline{AeA}$.

Let x be a positive element in $\overline{AeA}$ and $\{y_n\}$ be an increasing sequence in Ω_+ with $y_n \nearrow x$. Then

$$\phi(x) \leq \lim \phi(y_n) = \lim \psi(y_n) \leq \psi(x),$$

where the first inequality is because ϕ is lower semi-continuous.

Therefore, combined with $(*)$, we can get $\phi(x) = \psi(x)$ for all $x \in \overline{AeA}$.

Let ξ be a lower semi-continuous trace on A with $\phi(x) = \xi(x) < \infty$ for any $x \in (\overline{eAe})_+$, then by above

$$\begin{aligned} \xi(x) &= \sup\{\xi(y) : y \preccurlyeq x, y \in (eAe)_+\} \\ &= \sup\{\phi(y) : y \preccurlyeq x, y \in (eAe)_+\} \\ &= \psi(x) = \phi(x). \end{aligned}$$

□

Definition 2.2.6. *Let A, B be two C^*-algebras, we say that A and B have isomorphic Elliott Invariant if there are a scaled ordered isomorphism α from K_0A to K_0B and an affine isomorphism ξ from TB to TA such that α and ξ are compatible.*

Definition 2.2.7. *Let A, B be two C^*-algebras, we say that A and B have isomorphic Stevens' Invariant if*

(1) There exists a scaled ordered isomorphism $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$;

(2) For any $e \in \mathcal{P}(A)$, $f \in \mathcal{P}(B)$ with $\alpha[e] = [f]$, there exists an isomorphism $\xi^{e,f} : \mathrm{T}_\mathrm{F}(\overline{fBf}) \to \mathrm{T}_\mathrm{F}(\overline{eAe})$ such that for any $e' \leq e$, $f' \leq f$, $\alpha([e']) = [f']$, $\xi^{e,f}$ and $\xi^{e',f'}$ are compatible, i.e.

$$\begin{array}{ccc} \mathrm{T}_\mathrm{F}(\overline{fBf}) & \xrightarrow{\xi^{e,f}} & \mathrm{T}_\mathrm{F}(\overline{eAe}) \\ \downarrow i & & \downarrow i \\ \mathrm{T}_\mathrm{F}(\overline{f'Bf'}) & \xrightarrow{\xi^{e',f'}} & \mathrm{T}_\mathrm{F}(\overline{e'Ae'}) \end{array}$$

is commutative, where $\mathcal{P}(A)$ and $\mathcal{P}(B)$ denote the set of all projections in A and B respectively and i stands for the inclusion map defined by $i(\tau) = \tau|_{\overline{e'Ae'}}$ for any $\tau \in \mathrm{T}_\mathrm{F}(\overline{eAe})$.

Remark 2.2.8. Note that by the condition (2) of Definition 2.2.7, we can get that α and $\xi^{e,f}$ are compatible for each pair of projections $e \in \mathcal{P}(A)$, $f \in \mathcal{P}(B)$ satisfying $\alpha[e] = [f]$ (see 1.19 in [30]).

Theorem 2.2.9. *Let A, B be two C*-algebras with the ideal property. Suppose that A and B have isomorphic Elliott's Invariant, then A and B have isomorphic Stevens' Invariant.*

Proof. Let A, B be two C*-algebras with the ideal property. Suppose A and B have isomorphic Elliott's Invariant and let $\tau \in \mathrm{T_F}(\overline{fBf})$ be any finite trace, where f is a projection in B. Define a trace τ' on the closed ideal $\overline{BfB}$ by

$$\tau'(x) = \sup\{\tau(y) : y \preccurlyeq x, y \in (\overline{fBf})_+\}, \text{ for all } x \in \overline{BfB}.$$

Then by Theorem 2.1.7, τ' is lower semi-continuous. Let $\tilde{\tau}$ be a trace on B defined by

$$\tilde{\tau}(x) = \begin{cases} \tau'(x) & \text{if } x \in \overline{BfB} \\ \infty & \text{otherwise.} \end{cases}$$

Then $\tilde{\tau}$ is a lower semi-continuous trace on B. Since $\mathrm{T}A$ and $\mathrm{T}B$ are isomorphic, let $\tilde{\phi} = \xi(\tilde{\tau}), e = \alpha^{-1}(f)$, by Lemma 2.2.4, $\tilde{\phi}(x) = \infty$ for any $x \in A \setminus (\overline{AeA})$.

Let $\phi' = \tilde{\phi}|_{\overline{AeA}}$ and $\phi = \tilde{\phi}|_{\overline{eAe}}$. Therefore, we have found a finite trace ϕ.

Now if we have a trace $\phi = \tilde{\phi}|_{\overline{eAe}} \in \mathrm{T_F}(\overline{eAe})$, let ψ be a trace on $\overline{AeA}$ defined by

$$\psi(x) = \sup\{\phi(y) : y \preccurlyeq x, \ y \in \overline{eAe}\}.$$

Then by Theorem 2.1.7, ψ is a lower semi-continuous trace on $\overline{AeA}$, and it's obvious that $\phi'(x) = \psi(x)$ for all $x \in \overline{eAe}$. By Theorem 2.2.5, $\psi(x) = \phi'(x)$ for all $x \in \overline{AeA}$. Therefore, by the same way of finding ϕ above, we can find exactly $\tau \in \mathrm{T_F}(\overline{fBf})$.

Based on above, for any $\tau \in \mathrm{T_F}(\overline{fBf})$, there exists one and only one trace $\phi \in \mathrm{T_F}(\overline{eAe})$ corresponds to τ, and vise versa. Let us denote this map by $\xi^{e,f}$ and it is obvious that $\xi^{e,f}$ is an affine map since $\xi : \mathrm{T}B \to \mathrm{T}A$ is an affine map.

Let $e' \in \mathcal{P}(A)$ and $f' \in \mathcal{P}(B)$ be such that $e' \leq e, \ f' \leq f$ and $\alpha([e']) = [f']$. Let $\tau_f \in \mathrm{T_F}(\overline{fBf})$ be any finite trace. We need to show that $i \circ \xi^{e,f}(\tau_f) = \xi^{e',f'} \circ i(\tau_f)$ (see 2.2.7). Let $i(\tau_f) = \tau_{f'}$ and let $\widetilde{\tau}_f, \widetilde{\tau}_{f'} \in \mathrm{T}B$ be extensions of τ_f and $\tau_{f'}$ defined as above, respectively. Let $\widetilde{\phi}_e = \xi(\widetilde{\tau}_f)$ and $\widetilde{\phi}_{e'} = \xi(\widetilde{\tau}_{f'})$. By the definition of $\tau_{f'}$, we know that $\tau_{f'} = \tau_f|_{f'Bf'}$. Thus by Theorem 2.2.5, $\widetilde{\tau}_{f'} = \widetilde{\tau}_f$ on $\overline{Bf'B}$. And for $x \notin \overline{Bf'B}$, $\widetilde{\tau}_{f'}(x) = \infty$. Therefore, we have the following equality

$$t\widetilde{\tau}_f + (1-t)\widetilde{\tau}_{f'} = \widetilde{\tau}_{f'}, \text{ for any } 0 \leq t < 1.$$

Taking $t = 1/2$ and by the additivity of the map ξ, we get

$$\frac{1}{2}\xi(\widetilde{\tau}_f) + \frac{1}{2}\xi(\widetilde{\tau}_{f'}) = \xi(\widetilde{\tau}_{f'}),$$

which implies $(\widetilde{\phi_e})|_{e'Ae'} = (\widetilde{\phi_{e'}})|_{e'Ae'}$. Therefore,

$$i \circ \xi^{e,f}(\tau_f) = (\widetilde{\phi_e})|_{\overline{e'Ae'}} = (\widetilde{\phi_{e'}})|_{\overline{e'Ae'}} = \xi^{e',f'} \circ i(\tau_f).$$

Thus $\xi^{e,f}$ and $\xi^{e',f'}$ are compatible. Therefore, A and B have isomorphic Stevens' Invariant. □

The following lemma is from [38] (see Lemma 3.3.6 of [38]).

Lemma 2.2.10. *If p is a projection in A, $b \in A_+$ and p is in the ideal generated by b, then there are $x_1, x_2, \cdots, x_k \in A$ such that $p = \sum\limits_{i=1}^{k} x_i b x_i^*$.*

Theorem 2.2.11. *Let A, B be two C*-algebras with the ideal property. Suppose that A and B have isomorphic Stevens' Invariant, then A and B have isomorphic Elliott's Invariant.*

Proof. Let A, B be two C*-algebras with isomorphic Stevens' Invariant. For $\tau \in \mathrm{T}B$, we need to find $\phi \in \mathrm{T}A$ corresponds to τ.

Let $I_{fin\tau} = \{x \in B : \tau(x) < \infty\}$, which is an ideal of B. Let $\mathcal{P}_\tau := \{p \in \bar{I}_{fin\tau} : p \text{ is a projection}\}$. For any $f \in \mathcal{P}_\tau$, by Lemma 2.2.2, $\tau(f) < \infty$, and $\tau|_{\overline{fBf}} \triangleq \tau_f$ is a finite trace on $\overline{fBf}$. Let $e = \alpha^{-1}(f)$. Since A and B have isomorphic Stevens' Invariant, $\mathrm{T_F}(\overline{fBf})$ and $\mathrm{T_F}(\overline{eAe})$ are isomorphic, there exists $\phi_e \in \mathrm{T_F}(\overline{eAe})$ corresponding to τ_f.

First, we can extend ϕ_e on the closed ideal $\overline{AeA}$ by (still denoted by ϕ_e)

$$\phi_e(x) := \sup\{\phi_e(y) : y \preccurlyeq x,\ y \in (\overline{eAe})_+\}, \quad \text{for all } x \in \overline{AeA}.$$

Then define a lower semi-continuous trace ϕ on A by:

$$\phi(x) = \begin{cases} \phi_e(x) & \text{if } x \in \overline{AeA} \text{ for some } e \in \alpha^{-1}(\mathcal{P}_\tau), \\ \infty & \text{otherwise.} \end{cases}$$

Claim:ϕ is well-defined.
If $x \in (\overline{Ae_1A}) \cap (\overline{Ae_2A})$ with $e_i = \alpha^{-1}(f_i)$ for some $f_i \in \mathcal{P}_\tau$, $i = 1, 2$, then let $J_0 = (\overline{Ae_1A}) \cap (\overline{Ae_2A})$, which is a closed ideal of A. Since A is a C*-algebra with the ideal property, J_0 is generated by projections inside it. Let I_0 be a closed ideal of B generated by the set $\{q : \alpha^{-1}(q) \text{ is a projection in } J_0\}$. Then we have

$$I_0 = (\overline{Bf_1B}) \cap (\overline{Bf_2B}) \quad \text{and} \quad \tau_{f_1}|_{I_0} = \tau_{f_2}|_{I_0}.$$

Let p be any projection in J_0 and let $q = \alpha^{-1}(p)$. Then by Lemma 2.2.10, there exist natural numbers n_1, n_2 such that $p \le n_i e_i$ for $i = 1, 2$. Therefore, by the compatibility of Stevens' Invariant, following diagrams are commutative:

$$\begin{array}{ccc} \mathrm{T_F}(\overline{\widetilde{f}_1 B \widetilde{f}_1}) & \xrightarrow{\xi^{\widetilde{e}_1,\widetilde{f}_1}} & \mathrm{T_F}(\overline{\widetilde{e}_1 A \widetilde{e}_1}) \\ \downarrow i & & \downarrow i \\ \mathrm{T_F}(\overline{qBq}) & \xrightarrow{\xi^{p,q}} & \mathrm{T_F}(\overline{pAp}) \end{array},$$

$$\begin{array}{ccc}
\mathrm{T_F}(\overline{\widetilde{f}_2 B \widetilde{f}_2}) & \xrightarrow{\xi^{\widetilde{e}_2,\widetilde{f}_2}} & \mathrm{T_F}(\overline{\widetilde{e}_2 A \widetilde{e}_2}) \;, \\
\big\downarrow i & & \big\downarrow i \\
\mathrm{T_F}(\overline{qBq}) & \xrightarrow{\xi^{p,q}} & \mathrm{T_F}(\overline{pAp})
\end{array}$$

where $\widetilde{f}_i = n_i f$ and $\widetilde{e}_i = n_i e_i$ for $i = 1, 2$. Since

$$\mathrm{T_F}(\overline{\widetilde{f}_i B \widetilde{f}_i}) = \mathrm{T_F}(\mathrm{M}_{n_i}(\overline{f_i B f_i})) = \mathrm{T_F}(\overline{f_i B f_i}),$$

$$\mathrm{T_F}(\overline{\widetilde{e}_i A \widetilde{e}_i}) = \mathrm{T_F}(\mathrm{M}_{n_i}(\overline{e_i A e_i})) = \mathrm{T_F}(\overline{e_i A e_i}),$$

we can get the following commutative diagrams:

$$\begin{array}{ccc}
\mathrm{T_F}(\overline{f_1 B f_1}) & \xrightarrow{\xi^{e_1,f_1}} & \mathrm{T_F}(\overline{e_1 A e_1}) \;, \\
\big\downarrow i & & \big\downarrow i \\
\mathrm{T_F}(\overline{qBq}) & \xrightarrow{\xi^{p,q}} & \mathrm{T_F}(\overline{pAp})
\end{array}$$

$$\begin{array}{ccc}
\mathrm{T_F}(\overline{f_2 B f_2}) & \xrightarrow{\xi^{e_2,f_2}} & \mathrm{T_F}(\overline{e_2 A e_2}) \;. \\
\big\downarrow i & & \big\downarrow i \\
\mathrm{T_F}(\overline{qBq}) & \xrightarrow{\xi^{p,q}} & \mathrm{T_F}(\overline{pAp})
\end{array}$$

Therefore,

$$i \circ \xi^{e_1,f_1}(\tau_{f_1}) = \xi^{p,q} \circ i(\tau_{f_1}) \text{ which implies } \phi_{e_1}|_{\overline{pAp}} = \xi^{p,q}(\tau_{f_1}|_{\overline{qBq}}),$$

$$i \circ \xi^{e_2,f_2}(\tau_{f_2}) = \xi^{p,q} \circ i(\tau_{f_2}) \text{ which implies } \phi_{e_2}|_{\overline{pAp}} = \xi^{p,q}(\tau_{f_2}|_{\overline{qBq}}).$$

Since $\tau_{f_1}|_{\overline{qBq}} = \tau_{f_2}|_{\overline{qBq}}$, $\phi_{e_1}|_{\overline{pAp}} = \phi_{e_2}|_{\overline{pAp}}$. Then by Theorem 2.2.5 $\phi_{e_1}|_{\overline{ApA}} = \phi_{e_2}|_{\overline{ApA}}$. Therefore, $\phi_{e_1}|_{J_0} = \phi_{e_2}|_{J_0}$ since J_0 is generated by projections inside it. Thus, ϕ is well-defined.

If we starting with $\phi \in \mathrm{T}A$, as above process, we can find a lower semi-continuous trace in $\mathrm{T}B$, which is exactly τ. Therefore, $\mathrm{T}A$ and $\mathrm{T}B$ are isomorphic, which means A and B have isomorphic Elliott Invariant.

□

2.3 Cuntz Comparability

Let A be a C*-algebra, and let $\mathrm{M}_n(A)$ denote the $n \times n$ matrices whose entries are elements of A. Let $\mathrm{M}_\infty(A)$ denote the algebraic limit of the direct system $(\mathrm{M}_n(A), \phi_n)$, where $\phi_n : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A)$ is given by

$$a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathrm{M}_\infty(A)_+$ (resp. $\mathrm{M}_n(A)$) denote the positive elements in $\mathrm{M}_\infty(A)$ (resp. $\mathrm{M}_n(A)$). For positive elements a and b in $\mathrm{M}_\infty(A)$, write $a \oplus b$ to denote the element $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, which is also positive in $\mathrm{M}_\infty(A)$.

Definition 2.3.1. *Given $a, b \in M_\infty(A)_+$, we say that a is Cuntz subequivalent to b (written $a \precsim b$) if there is a sequence $(v_n)_{n=1}^\infty$ of elements of $M_\infty(A)$ such that*

$$\|v_n b v_n^* - a\| \xrightarrow{n\to\infty} 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. This relation is an equivalence relation, and write $\langle a \rangle$ for the equivalence class of a. The set

$$\mathrm{W}(A) := \mathrm{M}_\infty(A)_+ / \sim$$

becomes a positively ordered Abelian monoid when equipped with the operation

$$\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$$

and the partial order

$$\langle a \rangle \leq \langle b \rangle \Longleftrightarrow a \precsim b.$$

In the sequel, we refer to this object as the Cuntz semigroup of A.

Proposition 2.3.2. *([54], Proposition 2.2) Let A be a C^*-algebra. Let $a, p \in \mathrm{M}_\infty(A)_+$ be such that p is a projection and $p \precsim a$. Then there exists $b \in \mathrm{M}_\infty(A)_+$ such that $p \oplus b \sim a$.*

Definition 2.3.3. *Let A be a local C^*-algebra. A dimension function on A is a mapping $D : A \to [0, \infty)$ such that:*

(i) $\sup\{D(a) | a \in A\} = 1$ (normalization).

(ii) If $a \perp b$ (i.e., $ab = ab^* = a^*b = a^*b^* = 0$), then $D(a + b) = D(a) + D(b)$.

(iii) For all a, $D(a) = D(aa^*) = D(a^*a) = D(a^*)$.

(iv) If $0 \leq a \leq b$, then $D(a) \leq D(b)$.

(v) If $a \precsim b$ (i.e., there exists x_n, y_n with $\{x_n b y_n\}$ converging to a in norm), then $D(a) \leq D(b)$.

Proposition 2.3.4. *([57], Corollary 4.7) Let A be a C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be a $\mathcal{Z}$-absorbing C^*-algebra). Let a, b be positive elements in A. Suppose that a belongs to $\overline{AbA}$ and such that $d(a) < d(b)$ for every dimension function d on A with $d(b) = 1$. Then $a \precsim b$.*

Lemma 2.3.5. *Let A be an exact C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be a $\mathcal{Z}$-absorbing C^*-algebra). Let a, b be positive elements in A. Suppose that a belongs to $\overline{AbA}$ and such that $d_\tau(a) < d_\tau(b)$ for every dimension function d_τ on A with $d_\tau(b) = 1$. Then $a \precsim b$.*

Proof. Suppose d is any dimension function on A with $d(b) = 1$. Let

$$\bar{d}(\langle x \rangle) = \lim_{\varepsilon \to 0} d(\langle f_\varepsilon(x) \rangle),$$

where

$$f_\varepsilon(t) = \begin{cases} 0, & t \le \varepsilon \\ \frac{t-\varepsilon}{\varepsilon}, & \varepsilon \le t \le 2\varepsilon \\ 1, & t \ge 2\varepsilon. \end{cases}$$

Then $\bar{d}$ is a lower semi-continuous dimension function on A. Therefore, $\bar{d} = d_\tau$ for some $\tau \in \mathrm{T}A$. (This follows from Blackadar and Handelman, [[3], Theorem II.2.2].)

If $\bar{d}(b) = 0$, then $a \in \overline{AbA}$ implies $a \precsim b \otimes 1_k$ for some integer k. Therefore, $\bar{d}(a) = 0$ since $0 \le \bar{d}(a) \le k\bar{d}(b) = 0$. Hence $d(\langle f_\varepsilon(a) \rangle) \le \bar{d}_\tau(a) = 0$ for any $\varepsilon > 0$.

If $\bar{d}(b) \ne 0$, then let $l \triangleq \bar{d}/\bar{d}(b)$, which is a lower semi-continuous dimension function with $l(b) = 1$. Then by the assumption

$$d(\langle f_\varepsilon(a) \rangle)/\bar{d}(b) \le l(\langle a \rangle) < l(\langle b \rangle) \le d(\langle b \rangle)/\bar{d}(b).$$

Hence

$$d(\langle f_\varepsilon(a) \rangle) \le \bar{d}(\langle a \rangle) < \bar{d}(\langle b \rangle) \le d(\langle b \rangle).$$

In either case, we have

$$d(\langle f_\varepsilon(a) \rangle) < d(\langle b \rangle) \qquad \text{for } \forall \varepsilon > 0.$$

Since $a \in \overline{AbA}$, $f_\varepsilon(a) \in \overline{AbA}$. Therefore, by Proposition 2.3.4

$$f_\varepsilon(a) \precsim b \qquad \text{for } \forall \varepsilon > 0.$$

Hence $a \precsim b$. □

Definition 2.3.6. *Let A be a C^*-algebra with the ideal property. Let A_{++} be the set of A_+ consists of all positive elements which are not Cuntz equivalent to a non-zero projection in any quotient of A.*

Lemma 2.3.7. *Let A be an exact C^*-algebra for which $W(A)$ is almost unperforated (in particular, A could be a $\mathcal{Z}$-absorbing C^*-algebra). Let $a \in A_{++}$ and p be a projection in A. Then $p \precsim a$ if and only if $p \in \overline{AaA}$ and $d_\tau(p) < d_\tau(a)$ for each $\tau \in \mathrm{T}A$ with $d_\tau(a) = 1$.*

Proof. The reverse direction immediately follows from Lemma 2.3.5. Now suppose $p \precsim a$, then by Proposition 2.3.2, there exists a positive element c with $p \oplus c \sim a$. Considering the quotient $A/(c)$, where (c) stands for the ideal generated by c, we have $a \sim p$ in $A/(c)$. Therefore, $0 = a = p$ in $A/(c)$. Hence $p \in (c)$.

If $d_\tau(c) = 0$, then $d_\tau(p) = 0$, which implies $d_\tau(a) = 0$. Therefore, if $d_\tau(a) \ne 0$, then $d_\tau(c) \ne 0$, hence $d_\tau(p) < d_\tau(a)$. □

Chapter 3

C* exponential length

Let X be a compact Hausdorff space. In this paper, we give an example to show that there is $u \in \mathrm{C}(X) \otimes \mathrm{M}_n$ with $\det(u(x)) = 1$ for all $x \in X$ and $u \sim_h 1$ such that the C* exponential length of u (denoted by $cel(u)$) can not be controlled by π. Moreover, in simple inductive limit C*-algebras, similar examples also exist.

3.1 Preliminaries

First we recall some definitions and lemmas.

Definition 3.1.1. *Let X be a compact metric space and $B = C(X) \otimes M_n$. For $u \in U(B)$(unitary group of B), let $\det(u)$ be a function from X to S^1 whose value at x is $\det(u(x))$.*

Definition 3.1.2. *Let A be a unital C*-algebra and u be a unitary which lies in the connected component of the identity 1 in A. Define the C* exponential length of u (denoted by $cel(u)$) as follows:*

$$cel(u) = \inf\{\sum_{i=1}^{k} \|h_i\| : u = exp(ih_1)exp(ih_2)...exp(ih_k)\}.$$

Remark 3.1.3. For a unital C*-algebra A, let $U_0(A)$ be the connected component of $U(A)$ containing the identity 1. Recall from [55] that if $u \in U_0(A)$, then the C* exponential length $cel(u)$ is equal to the infimum of the lengths of rectifiable paths from u to 1 in $U(A)$.

The following lemma is an easy example for calculating the C* exponential length.

Lemma 3.1.4. *Let $\alpha \in \mathbb{R}$ and $u \in C[0,1]$ be defined by $u(t) = \exp(it\alpha)$. Then*

$$cel(u) = \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi|.$$

Moreover, if $|\alpha| \leq 2\pi$, then $cel(u) = |\alpha|$.

Proof. Since $cel(u) = \inf\{\text{length}(u_s) : u_s \text{ is a path in } U(\text{C}[0,1]) \text{ from } u \text{ to } 1\}$, let $v_s(t)$ be any path from 1 to u, that is, $v_0(t) = 1, v_1(t) = u(t)$. Without loss of generality, we can assume v_s is piecewise smooth. Then $\text{length}(v_s) = \int_0^1 \|\frac{dv}{ds}\| ds$. Since $v_s(t)$ can be considered as a map from $[0,1] \times [0,1]$ to S^1 and $\mathbb{R}$ is a covering space of S^1, there exists a unique map $\tilde{v}_s(t)$ from $[0,1] \times [0,1]$ to $\mathbb{R}$ such that:

$$v_s(t) = \pi(\tilde{v}_s(t)) \text{ and } \widetilde{v}_0(0) = 0, \qquad (*)$$

where $\pi(x) = e^{ix}$. Therefore

$$\frac{dv}{ds} = \pi'(\tilde{v}_s(t)) \cdot \frac{d\widetilde{v}}{ds},$$

which implies $\|\frac{dv}{ds}\| = \|\frac{d\tilde{v}}{ds}\|$.

By $(*)$, $\pi(\tilde{v}_0(t)) = v_0(t) = 1$, hence $\tilde{v}_0(t) \in 2\pi\mathbb{Z}$ for all $t \in [0,1]$. Since $\tilde{v}_0(0) = 0$ and $\tilde{v}_0(t)$ is continuous, $\tilde{v}_0(t) = 0$ for all $t \in [0,1]$. In addition, by $(*)$, we can also get $\pi(\tilde{v}_1(t)) = v_1(t) = \exp(it\alpha)$. Thus $\tilde{v}_1(t) - \alpha t \in 2\pi\mathbb{Z}$ for all t. By continuity of $\tilde{v}_1(t) - \alpha t$, there exists some integer k such that $\tilde{v}_1(t) - \alpha t = 2k\pi$ for all $t \in [0,1]$. Therefore,

$$\int_0^1 \|\frac{d\tilde{v}}{ds}\| ds \geq \| \int_0^1 \frac{d\tilde{v}}{ds} ds\| = \max_{t\in[0,1]} |\tilde{v}_1(t) - \tilde{v}_0(t)| \geq \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi|.$$

Let $L = \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi|$ and $k_0 \in \mathbb{Z}$ be such that $L = \max_{t\in[0,1]} |\alpha t - 2k_0\pi|$. Fix

$$v_s(t) = \exp\{is(\alpha t - 2k_0\pi)\},$$

then $v_0(t) = \exp\{0\} = 1$ and $v_1(t) = \exp\{i\alpha t - 2k_0\pi i\} = \exp\{i\alpha t\}$ and

$$\int_0^1 ||\frac{dv}{ds}|| ds = \int_0^1 ||\alpha t - 2k_0\pi|| ds = \int_0^1 \max_{t\in[0,1]} |\alpha t - 2k_0\pi| ds = \int_0^1 L ds = L.$$

Thus $v_s(t)$ is a path in $U(C[0,1])$ connecting 1 and $u(t)$ with length L. Therefore, $cel(u) = L$.

Let us assume $|\alpha| \leq 2\pi$. For $k = 0$,

$$\max_{t\in[0,1]} |\alpha t - 2k\pi| = \max_{t\in[0,1]} |\alpha t - 0| = |\alpha|.$$

For $k \neq 0$,

$$\max_{t\in[0,1]} |\alpha t - 2k\pi| \geq |0 - 2k\pi| = 2|k|\pi \geq |\alpha|.$$

Hence, $\min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi| = |\alpha|$. That is $cel(u) = |\alpha|$. □

3.2 Counterexamples

Lemma 3.2.1. *(see Lemma 2.4 of [46]) The set of elements in $SU(M_n(\mathbb{C}))$ with at least one repeated eigenvalue is the union of finitely many submanifolds of $SU(M_n(\mathbb{C}))$, all of codimension at least 3.*
(Here $SU(M_n(\mathbb{C}))$ is the set of elements in $U(M_n(\mathbb{C}))$ with determinant 1.)

Corollary 3.2.2. *Let $Z = \{u \in U(M_n(\mathbb{C})) : u$ has repeated eigenvalues$\}$. Then Z is the union of finitely many submanifolds of $U(M_n(\mathbb{C}))$, all of codimension at least 3.*

Proof. We use the way of the proof of Lemma 3.2.1(see page 136-137 of [46]). For the convenience of the reader, we repeat the proof for our case.

Let P be a partition of n, that is, a sequence $(n_1, \cdots, n_k)$ of positive integers such that $n_1+\cdots+n_k = n$ and $n_1 \geq n_2 \geq \cdots \geq n_k$. Let M_P be the set of all $u \in U(\mathrm{M}_n(\mathbb{C}))$ having exactly k distinct eigenvalues, with multiplicities $n_1, \cdots, n_k$. Let G_P be the set of sequences $(V_1, \cdots, V_k)$ of orthogonal subspaces of $\mathbb{C}^n$ such that $\dim(V_j) = n_j$ for each j. Let W_P be the set of k-tuples of distinct elements $(\lambda_1, \cdots, \lambda_k) \in (S^1)^k$, where S^1 is the unit circle on the complex plane. Then W_P and G_P are smooth manifolds. Define $f_P : G_P \times W_P \to M_P$ by sending $(V_1, \cdots, V_k, \lambda_1, \cdots, \lambda_k)$ to the unitary $u \in U(\mathrm{M}_n(\mathbb{C}))$ such that $u\xi = \lambda_j\xi$ for $\xi \in V_j$. Then f_P is a smooth surjective local homeomorphism from $G_P \times W_P$ to M_P.

To show that M_P is a smooth manifold, we must show that f_P is a local diffeomorphism, that is for each $x \in G_P \times W_P$ there is a smooth map g from a neighborhood of $f_P(x)$ in $U(\mathrm{M}_n(\mathbb{C}))$ to $G_P \times W_P$ such that $g \circ f_P$ is the identity near x and $f_P \circ g$ is the identity on a neighborhood of $f_P(x)$ in M_P. To construct g, let $x = (V_1, \cdots, V_k, \lambda_1, \cdots, \lambda_k)$ and let $u = f_P(x)$. Choose $\varepsilon > 0$ such that the ε-disks about $\lambda_1, \cdots, \lambda_k$ in $\mathbb{C}$ are disjoint. For v close enough to u, let p_j be the spectral projection corresponding to $\{\lambda \in \mathbb{C} : |\lambda - \lambda_j| < \varepsilon\}$ and W_j be the corresponding subspace. Let $\mu_j = \det(p_j v p_j)^{1/n_j}$, where $p_j v p_j$ is regarded as an operator on W_j and the n_jth root is the branch going through λ_j. Then $g(v) = (W_1, \cdots, W_k, \mu_1, \cdots, \mu_k)$ will do. (Note that it is smooth because the projections p_j can be obtained via holomorphic functional calculus.)

$U(\mathrm{M}_n(\mathbb{C}))$ is the disjoint union of the manifolds M_P as P runs through all partitions. So the corollary is proved if we can show that $codim(M_P) \geq 3$ for $P \neq (1, \cdots, 1)$. It is easily seen that the map g above extends to a local diffeomorphism $v \mapsto (W_1, \cdots, W_k, \mu_1, \cdots, \mu_k, \mu_1^{-1}p_1vp_1, \cdots, \mu_k^{-1}p_kvp_k)$ to a manifold locally diffeomorphic to $G_P \times W_P \times SU(\mathrm{M}_{n_1}(\mathbb{C})) \times \cdots \times SU(\mathrm{M}_{n_k}(\mathbb{C}))$, and the dimension of the last part is at least 3 if some $n_j \neq 1$.

□

Lemma 3.2.3. *Let $f(s,t) : X \triangleq [0,1] \times [0,1] \to U(M_n(\mathbb{C}))$ be a smooth map. For any $\delta > 0$, there is a smooth map $g(s,t) : [0,1] \times [0,1] \to U(M_n(\mathbb{C}))$ such that:*

(1) $\|f-g\|<\delta$, $\|\frac{\partial f}{\partial s}(s,t)-\frac{\partial g}{\partial s}(s,t)\|<\delta$, $\|\frac{\partial f}{\partial t}(s,t)-\frac{\partial g}{\partial t}(s,t)\|<\delta$;
(2) $g(s,t)$ *has no repeated eigenvalues for all* $(s,t)\in[0,1]\times[0,1]$.

Proof. This is a standard transversal argument. See, for example, [28] page 70-71. We note that though the statement in [28] does not assert that the derivatives are close, the proof really shows that. For the convenience of the reader, we repeat the construction here for our special case.

By smoothly extending f to an open neighborhood of $[0,1]\times[0,1]$, we can assume f is defined on an open manifold without boundary. Let Z be a subspace of $U(\mathrm{M}_n(\mathbb{C}))$ defined by

$$Z=\{u\in U(\mathrm{M}_n(\mathbb{C})):\ u \text{ has repeated eigenvalues}\}.$$

Since $U(\mathrm{M}_n(\mathbb{C}))$ is a subspace of $\mathrm{M}_n(\mathbb{C})$ and the latter can be identified with $\mathbb{R}^{2n^2}$ as a topological space, f is a smooth map from X to $\mathbb{R}^{2n^2}$. Let B be the open unit ball of $\mathbb{R}^{2n^2}$ (with Euclidean metric). Then B corresponds to some open ball (contained in the unit ball of $\mathrm{M}_n(\mathbb{C})$) in $\mathrm{M}_n(\mathbb{C})$ with the matrix norm, for which we still use the notation B. Let $0<\varepsilon<1/2$. For $x\in X$, $r\in B$, define

$$F(x,r)=\pi[f(x)+\varepsilon r],$$

where $\pi: Gl_n(\mathbb{C})\to U(\mathrm{M}_n(\mathbb{C}))$ is defined by the Polar decomposition, which serves as the map π in [28] page 69 from the tubular neighborhood of $U(\mathrm{M}_n(\mathbb{C}))$ (which is Y in the notation of [28] page 70) to $U(\mathrm{M}_n(\mathbb{C}))$. Notice that $Gl_n(\mathbb{C})$ is an open submanifold of $\mathrm{M}_n(\mathbb{C})$. π can be considered as a map from a submanifold of $\mathbb{R}^{2n^2}$ to $\mathbb{R}^{2n^2}$. Even though all the scalars here are complex, the objects are being viewed as real manifolds. Note that for $A\in Gl_n(\mathbb{C})$, $\pi(A)$ is obtained by the Gram-Schmidt process. From the Gram-Schmidt process, one easily see that π is C^∞. Hence

$$F:[0,1]\times[0,1]\times B\ (\subseteq\mathbb{R}^{2n^2+2})\to\mathbb{R}^{2n^2}$$

is a C^∞ map.

Define $f_r: X\to U(\mathrm{M}_n(\mathbb{C}))$ by

$$f_r(x)=F(x,r).$$

Since π restricts to the identity on $U(\mathrm{M}_n(\mathbb{C}))$,

$$f_0(x)=F(x,0)=\pi(f(x))=f(x).$$

For fixed x, $r\to f(x)+\varepsilon r$ is certainly a submersion of $B\to\mathrm{M}_n(\mathbb{C})$. As the composition of two submersions is another, $r\to F(x,r)$ is a submersion. Therefore, F is transversal to Z. By Corollary 3.2.2, Z is a finite union of submanifolds of $U(\mathrm{M}_n(\mathbb{C}))$, say $\{N_1,\cdots,N_L\}$. So F is transversal to each N_j. Then by applying the

Transversality Theorem (see page 68 of [28]), we have that f_r is transversal to N_j for all $j = 1, \cdots, L$ and for almost all $r \in B$. Since each N_j is of codimension at least 3,

$$\dim(X) + \dim(N_j) < \dim(U(\mathrm{M}_n(\mathbb{C}))).$$

So f_r transversal to N_j implies $\mathrm{Im} f_r \bigcap N_j = \emptyset$. Therefore, $\mathrm{Im} f_r \bigcap Z = \emptyset$ for almost all $r \in B$.

Since F is smooth, and therefore $\frac{\partial F}{\partial s}$ is continuous with respect to r, for any $\delta > 0$, there exists $\eta > 0$ such that for all r with $\|r\| \le \eta$ we have

$$||\frac{\partial F}{\partial s}(s,t,r) - \frac{\partial F}{\partial s}(s,t,0)|| \le \delta, \quad ||\frac{\partial F}{\partial t}(s,t,r) - \frac{\partial F}{\partial t}(s,t,0)|| \le \delta.$$

Thus

$$||\frac{\partial f_r}{\partial s}(s,t) - \frac{\partial f}{\partial s}(s,t)|| = ||\frac{\partial F}{\partial s}(s,t,r) - \frac{\partial F}{\partial s}(s,t,0)|| \le \delta,$$
$$||\frac{\partial f_r}{\partial t}(s,t) - \frac{\partial f}{\partial t}(s,t)|| = ||\frac{\partial F}{\partial t}(s,t,r) - \frac{\partial F}{\partial t}(s,t,0)|| \le \delta.$$

Finally by taking r appropriately, we can get f_r satisfies the properties (1) and (2). Let $g = f_r$ and this completes the proof.

□

Corollary 3.2.4. *Let $\widetilde{F}_s$ be a rectifiable path in $U(M_k(C[0,1]))$. For any $\varepsilon > 0$, there exists a path F_s in $U(M_k(C[0,1]))$ such that:*
(1) $\|F - \widetilde{F}\| < \varepsilon$;
(2) $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$;
(3) $|length(\widetilde{F}) - length(F)| < \varepsilon$.
Moreover, if for each $t \in [0,1]$, $\widetilde{F}_1(t)$ has no repeated eigenvalues, then F can be chosen to be such that $F_1(t) = \widetilde{F}_1(t)$ for all $t \in [0,1]$.

Proof. Let ε_1 be a small number to be determined later. Let δ_0 be such that $|1-e^{i\theta}| \le \delta_0$ implies $|\theta| \le (1+\varepsilon_1)|1-e^{i\theta}|$ for $\theta \in \mathbb{R}$. For $\varepsilon > 0$, let $\delta = \min\{\delta_0, \varepsilon/6, 1/2\}$. By the definition of the length, there exist $0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$ such that

$$\|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| < \delta/2, \text{ for } j = 0, 1, \cdots, n-1,$$

and

$$\sum_{j=0}^{n-1} \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| \le length(\widetilde{F}_s) \le \sum_{j=0}^{n-1} \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| + \varepsilon/4.$$

Note that for each j, $\widetilde{F}_{s_j}(t)$ is a continuous map from $[0,1]$ to $U(\mathrm{M}_k(\mathbb{C}))$. There exist smooth maps $G_{s_j}(t) : [0,1] \to U(\mathrm{M}_k(\mathbb{C}))$, such that

$$\|G_{s_j} - \widetilde{F}_{s_j}\| = \sup_{t\in[0,1]} \|G_{s_j}(t) - \widetilde{F}_{s_j}(t)\| < \frac{\delta}{4n}, \qquad j = 0, 1, \cdots, n.$$

Then

$$\begin{aligned}\|G_{s_{j+1}} - G_{s_j}\| &= \|G_{s_{j+1}} - \widetilde{F}_{s_{j+1}} + \widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j} + \widetilde{F}_{s_j} - G_{s_j}\| \\ &\leq \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| + \frac{\delta}{2n} \leq \delta.\end{aligned}$$

And by the first equality, we can also get

$$\|G_{s_{j+1}} - G_{s_j}\| \geq \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| - \frac{\delta}{2n}.$$

Therefore,

$$\begin{aligned}|\|G_{s_{j+1}} - G_{s_j}\| - \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\|| &\leq \frac{\delta}{2n}, \\ |\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| - \sum_{j=0}^{n-1} \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\|| &\leq \frac{\delta}{2}.\end{aligned}$$

Thus

$$\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| - \frac{\delta}{2} \leq \text{length}(\widetilde{F}_s) \leq \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| + \frac{\delta}{2} + \varepsilon/4,$$

$$\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| - \frac{\varepsilon}{8} \leq \text{length}(\widetilde{F}_s) \leq \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| + \varepsilon/2. \quad (*)$$

Now we want to define a smooth function

$$\widetilde{G}_s(t) : [0,1] \times [0,1] \to U(\mathrm{M}_k(\mathbb{C}))$$

such that $\widetilde{G}_{s_j}(t) = G_{s_j}(t)$ for $j = 0, 1, \cdots, n,\ t \in [0,1]$. We will define it piece by piece on each subinterval $[s_j, s_{j+1}]$ $(j = 0, 1, \cdots, n-1)$.

Suppose

$$G_{s_j}^* G_{s_{j+1}} = U_j \begin{pmatrix} e^{i\alpha_1(t)} & 0 & \cdots & 0 \\ 0 & e^{i\alpha_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\alpha_k(t)} \end{pmatrix} U_j^*.$$

Since $\|G_{s_j}^* G_{s_{j+1}} - I\| = \|G_{s_j} - G_{s_{j+1}}\| \leq \delta < 1$, there exists a self-adjoint element $H_j(t)$ in $\mathrm{M}_k(C[0,1])$ such that $G_{s_j}^* G_{s_{j+1}}(t) = e^{iH_j(t)}$, (here $H_j(t) = -i \log[G_{s_j}^*(t) G_{s_{j+1}}(t)]$ which is a smooth function). Define

$$\widetilde{G}_s(t) = G_{s_j}(t) e^{i\frac{s-s_j}{s_{j+1}-s_j} H_j} \text{ for } s_j \leq s \leq s_{j+1},\ t \in [0,1],\ j = 0, 1, \cdots, n-1.$$

Then $\widetilde{G}_s(t)$ $(s_j \le s \le s_{j+1})$ is a path in $U(\mathrm{M}_k(C[0,1]))$ from G_{s_j} to $G_{s_{j+1}}$ and $\widetilde{G}_s(t)$ is smooth for $(s,t) \in [s_j, s_{j+1}] \times [0,1]$. Moreover,

$$
\begin{aligned}
&\mathrm{length}(\widetilde{G}_s|_{s_j \le s \le s_{j+1}}) \\
=&\int_{s_j}^{s_{j+1}} \|\frac{\partial \widetilde{G}_s}{\partial s}\| ds \\
\le&\int_{s_j}^{s_{j+1}} \frac{1}{s_{j+1}-s_j} \|G_{s_j}(t)H_j(t)\| ds \\
\le&(1+\varepsilon_1)\|G_{s_j}U_j \begin{pmatrix} 1-e^{i\alpha_1(t)} & 0 & \cdots & 0 \\ 0 & 1-e^{i\alpha_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-e^{i\alpha_k(t)} \end{pmatrix} U_j^*\| \\
=&(1+\varepsilon_1)\|G_{s_j}[I - U_j \begin{pmatrix} e^{i\alpha_1(t)} & 0 & \cdots & 0 \\ 0 & e^{i\alpha_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\alpha_k(t)} \end{pmatrix} U_j^*]\| \\
=&(1+\varepsilon_1)\|G_{s_j}[I - G_{s_j}^* G_{s_{j+1}}]\| \\
=&(1+\varepsilon_1)\|G_{s_j} - G_{s_{j+1}}\|.
\end{aligned}
$$

Therefore, $\widetilde{G}_s$ $(0 \le s \le 1)$ is a piecewise smooth path in $U(\mathrm{M}_k(C[0,1]))$ and

$$\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| \le \mathrm{length}(\widetilde{G}_s) \le (1+\varepsilon_1) \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\|.$$

Thus by $(*)$, we have

$$\mathrm{length}(\widetilde{G}_s)(1+\varepsilon_1)^{-1} - \frac{\varepsilon}{8} \le \mathrm{length}(\widetilde{F}_s) \le \mathrm{length}(\widetilde{G}_s) + \varepsilon/2.$$

Finally, pick any smooth monotone function $\xi : [0,1] \to [0,1]$ with

$$\xi(0) = 0,\ \xi(1) = 1,\ \frac{d^n \xi}{ds^n} |_{s=0} = 0,\ \frac{d^n \xi}{ds^n} |_{s=1} = 0 \ \text{ for all } n \ge 1.$$

Let

$$\widetilde{G}'_s(t) = G_{s_j}(t) e^{i\xi(\frac{s-s_j}{s_{j+1}-s_j})H_j} \text{ for } s_j \le s \le s_{j+1},\ t \in [0,1],\ j = 0, 1, \cdots, n-1.$$

Then $\widetilde{G}'_s(t)$ is smooth for all $(s,t) \in [0,1] \times [0,1]$ (since $\frac{\partial \widetilde{G}'_s(t)}{\partial s}|_{s=s_j} = 0$ from both left and right for all $j = 1, 2, \cdots, n-1$) and

$$\mathrm{length}(\widetilde{G}'_s) = \mathrm{length}(\widetilde{G}_s).$$

And for each $(s,t) \in [0,1] \times [0,1]$,

$$\begin{aligned}\|\widetilde{G}'_s(t) - \widetilde{F}_s(t)\| =& \|\widetilde{G}'_s(t) - \widetilde{G}'_{s_j}(t) + \widetilde{G}'_{s_j}(t) - \widetilde{F}_{s_j}(t) + \widetilde{F}_{s_j}(t) - \widetilde{F}_s(t)\| \\ \leq& \|\widetilde{G}_{s_{j+1}}(t) - \widetilde{G}_{s_j}(t)\| + \|\widetilde{G}_{s_j}(t) - \widetilde{F}_{s_j}(t)\| + \|\widetilde{F}_{s_j}(t) - \widetilde{F}_s(t)\| \\ \leq& \delta + \frac{\delta}{4n} + \frac{\varepsilon}{4} \leq \varepsilon/2,\end{aligned}$$

where s_j satisfies $s_j \leq s \leq s_{j+1}$.

Thus, by choosing ε_1 appropriately, we have

$$|\text{length}(\widetilde{G}'_s) - \text{length}(\widetilde{F}_s)| < \varepsilon/2 \text{ and } \|\widetilde{G}' - \widetilde{F}\| < \varepsilon/2.$$

Since $\widetilde{G}'_s$ can be seen as a smooth map from $[0,1]\times[0,1]$ to $U(\mathrm{M}_k(\mathbb{C}))$, by Lemma 3.2.3, there exists F such that $\|F - \widetilde{G}'\| < \varepsilon/2$ and $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$. Moreover,

$$|\text{length}(F_s) - \text{length}(\widetilde{G}'_s)| = |\int_0^1 \|\frac{\partial F}{\partial s}\| ds - \int_0^1 \|\frac{\partial \widetilde{G}'}{\partial s}\| ds| < \varepsilon/2.$$

Thus F satisfies properties (1)-(3), which is what we want.

Moreover, if $\widetilde{F}_1(t)$ has no repeated eigenvalues for all $t \in [0,1]$, then there exists $\eta > 0$ such that $\|u(t) - \widetilde{F}_1(t)\| < \eta$ implies $u(t)$ has no repeated eigenvalues for all $t \in [0,1]$. For $\varepsilon = \eta/2$, by the first part of the statement we can find a path F_s satisfying properties (1), (2), (3). Let $s_0 \in [s_{n-1}, 1)$ be such that $\|F_{s_0}(t) - F_1(t)\| \leq \eta/4$, (where s_{n-1} is a point of the partition of $[0,1]$ for which we mentioned in the proof of the first part of the statement). Then

$$\|F_{s_0}(t) - \widetilde{F}_1(t)\| = \|F_{s_0}(t) - F_1(t) + F_1(t) - \widetilde{F}_1(t)\| \leq 3\eta/4.$$

Now let us redefine $F_s(t)$ on the subinterval $[s_0, 1]$ (still use the notation $F_s(t)$) by a similar way as above:

$$F_s(t) = F_{s_0}(t) e^{i\frac{s-s_0}{1-s_0} H_n(t)}, \quad \text{for } s_0 \leq s \leq 1,$$

where $H_n(t) = -i \log[F^*_{s_0}(t)\widetilde{F}_1(t)]$. Since this newly defined path F_s lies in the η neighborhood of F_1, $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$. Thus this F_s is what we want.

□

Definition 3.2.5. *For a metric space (Y, d), let*

$$P^k Y := \{(y_1, y_2, ..., y_k) : y_i \in Y\}/\sim,$$

where $(y_1, y_2, ..., y_k) \sim (\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_k)$ if $\exists \sigma \in S_k$ such that $y_{\sigma(i)} = \tilde{y}_i$ for all $1 \leq i \leq k$. Let $[y_1, y_2, ..., y_k]$ denote the equivalent class of $(y_1, y_2, \cdots, y_k)$ in $P^k Y$. Define also the metric of $P^k Y$ as:

$$dist([y_1, y_2, ..., y_k], [\widetilde{y}_1, \widetilde{y}_2, ..., \widetilde{y}_k]) = \min_{\sigma \in S_k} \max_{1 \leq i \leq k} d(y_i, \widetilde{y}_{\sigma(i)}).$$

Lemma 3.2.6. *Let (Y, d) be a metric space and $\pi : \underbrace{Y \times Y \times ... \times Y}_{k} \longrightarrow P^kY$ be the quotient map. Let $X \subset \underbrace{Y \times Y \times ... \times Y}_{k}$ be the set consisting of those elements $(y_1, y_2, ..., y_k)$ with $y_i \neq y_j$ if $i \neq j$. Then the restriction of π to X is a covering map.*

Proof. Let $[z_1, z_2, ..., z_k] \in \pi(X)$ be a fixed point, by the definition of X, $z_i \neq z_j$ for $i \neq j$. Let

$$\varepsilon = \frac{1}{2} \min\{d(z_i, z_j) : i \neq j, 1 \leq i, j \leq k\},$$

$$N_\varepsilon = \{[y_1, y_2, ..., y_k] \in P^kY : \mathrm{dist}([y_1, y_2, ..., y_k], [z_1, z_2, ..., z_k]) < \varepsilon\}.$$

Claim: $N_\varepsilon \subseteq \pi(X)$.
In fact, if there is $[y_1, y_2, ..., y_k] \in N_\varepsilon$ and $y_i = y_j$ for some $1 \leq i, j \leq k$ and $i \neq j$, then $\exists \sigma \in S_k$ such that $d(y_i, z_{\sigma(i)}) < \varepsilon$ and $d(y_j, z_{\sigma(j)}) < \varepsilon$. Therefore

$$d(z_{\sigma(i)}, z_{\sigma(j)}) \leq d(y_i, z_{\sigma(i)}) + d(y_j, z_{\sigma(j)}) < 2\varepsilon,$$

which contradicts to the definition of ε.

It is easy to see that N_ε is an open neighbourhood of $[z_1, z_2, ..., z_k]$ and $\pi^{-1}(N_\varepsilon)$ consists of $k!$ pairwise disjoint open subsets (denoted by $U_i, 1 \leq i \leq k!$) of X, and the restriction of π to each U_i is a homeomorphism from U_i to N_ε. Hence, the restriction of π on X is a covering map. □

Lemma 3.2.7. *Let $F : [0, 1] \times [0, 1] \to P^kS^1$ be a continuous function, suppose*

$$F(s, t) = [x_1(s, t), x_2(s, t), ..., x_k(s, t)],$$

and for all $(s, t) \in [0, 1] \times [0, 1]$, $x_i(s, t) \neq x_j(s, t)$ if $i \neq j$. Then there are continuous functions $f_1, f_2, ..., f_k : [0, 1] \times [0, 1] \to S^1$ such that:

$$F(s, t) = [f_1(s, t), f_2(s, t), ..., f_k(s, t)].$$

Proof. Let $\pi : \underbrace{S^1 \times S^1 \times ... \times S^1}_{k} \to P^kS^1$ denote the quotient map, and let $X \subset \underbrace{S^1 \times S^1 \times ... \times S^1}_{k}$ be the set consisting of those elements $(x_1, x_2, \cdots, x_k)$ with $x_i \neq x_j$ if $i \neq j$. Then by Lemma 3.2.6 $\pi|_X$ is a covering map from X to $\pi(X)$ (which is a subset of P^kS^1).

Note from the assumption of the Lemma, the image of F is contained in $\pi(X)$. Since $[0, 1] \times [0, 1]$ is simply connected, by the standard lifting theorem for covering spaces, the map $F : [0, 1] \times [0, 1] \to \pi(X) \subset P^kS^1$ can be lifted to a map $F_1 : [0, 1] \times [0, 1] \to X(\subset \underbrace{S^1 \times S^1 \times ... \times S^1}_{k})$.

Let $\pi_j : S^1 \times S^1 \times ... \times S^1 \to S^1$ be the projection onto the jth coordinate. For $1 \le j \le k$, define functions $f_j : [0,1] \times [0,1] \to S^1$ by

$$f_j(s,t) = \pi_j(F_1(s,t)).$$

Then it is easy to see that f_j's satisfy the requirements. □

Remark 3.2.8. Let F_s be a path in $U(M_k(C[0,1]))$ such that $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$. Let $\Lambda : [0,1] \times [0,1] \to P_k S^1$ be the eigenvalue map of $F_s(t)$, i.e. $\Lambda(s,t) = [x_1(s,t), x_2(s,t), \cdots, x_k(s,t)]$, where $\{x_i(s,t)\}_{i=1}^k$ are eigenvalues of the matrix $F_s(t)$. By Lemma 3.2.7, there are continuous functions $f_1, f_2, ..., f_k : [0,1] \times [0,1] \to S^1$ such that:

$$\Lambda(s,t) = [f_1(s,t), f_2(s,t), ..., f_k(s,t)].$$

For each fixed $(s,t) \in [0,1] \times [0,1]$, there is a unitary $U_s(t)$ such that

$$F_s(t) = U_s(t) diag[f_1(s,t), f_2(s,t), ..., f_k(s,t)] U_s(t)^*.$$

Note that $U_s(t)$ can be chosen to be continuous, but in this paper we don't need this property.

Proposition 3.2.9. *If $U, V \in M_n(\mathbb{C})$ are unitaries with eigenvalues $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ respectively, then*

$$\min_{\sigma \in S_n} \max_i |u_i - v_{\sigma(i)}| \le \|U - V\|.$$

(The same result for a pair of Hermitian matrices is due to H. Weyl, called Weyl's Inequality see [65]).

Proof. See line 13-line 18 of page 71 of [2]. □

Lemma 3.2.10. *Let F_s be a path in $U(M_n(C[0,1]))$ and $f_s^1(t), f_s^2(t), ..., f_s^n(t)$ be continuous functions such that*

$$F_s(t) = U_s(t) diag[f_s^1(t), f_s^2(t), ..., f_s^n(t)] U_s(t)^*,$$

where $U_s(t)$ are unitaries. Suppose for any $(s,t) \in [0,1] \times [0,1]$, $f_s^i(t) \ne f_s^j(t)$ if $i \ne j$, then

$$length(F_s) \ge \max_{1 \le j \le n} \{length(f_s^j)\}.$$

(In this lemma, we assume that $F_s(t)$ is continuous, but we do not assume $U_s(t)$ is continuous.)

Proof. Let

$$\varepsilon = \min\{|(f_s^i(t) - f_s^j(t))| : \ i \neq j, \ 1 \leq i, j \leq n, \ s \in [0,1], \ t \in [0,1]\}.$$

Since for each j, $f_s^j(t)$ is continuous with respect to s, there exists $\delta > 0$ such that: for any partition $\mathcal{P} = \{s_1, s_2, \cdots, s_\lambda\}$ with $|\mathcal{P}| < \delta$,

$$\|f_{s_i}^j(t) - f_{s_{i-1}}^j(t)\| < \varepsilon/2 \quad \text{for all } 2 \leq i \leq \lambda, \quad 1 \leq j \leq n.$$

Then by Proposition 3.2.9,

$$\begin{aligned} \text{length}(F_s) &\geq \sum_{i=2}^{\lambda} \|F_{s_i} - F_{s_{i-1}}\| = \sum_{i=2}^{\lambda} \sup_{t\in[0,1]} \|F_{s_i}(t) - F_{s_{i-1}}(t)\| \\ &\geq \sum_{i=2}^{\lambda} \sup_{t\in[0,1]} [\min_{\sigma\in S_n} \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)|]. \end{aligned}$$

If $\sigma(j) \neq j$, then

$$\begin{aligned} &|f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)| \\ \geq &|f_{s_i}^j(t) - f_{s_i}^{\sigma(j)}(t)| - |f_{s_i}^{\sigma(j)}(t) - f_{s_{i-1}}^{\sigma(j)}(t)| \\ > &\varepsilon - \varepsilon/2 = \varepsilon/2. \end{aligned}$$

If $\sigma(j) = j$, then $|f_{s_i}^j(t) - f_{s_{i-1}}^j(t)| < \varepsilon/2$. Therefore,

$$\min_{\sigma\in S_n} \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)| = \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^j(t)|.$$

$$\begin{aligned} \text{length}(F_s) &\geq \sum_{i=2}^{\lambda} \sup_{t\in[0,1]} [\min_{\sigma\in S_n} \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)|] \\ &\geq \sum_{i=2}^{\lambda} \sup_{t\in[0,1]} \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^j(t)| \\ &= \sum_{i=2}^{\lambda} \max_{1\leq j\leq n} \|f_{s_i}^j(t) - f_{s_{i-1}}^j(t)\|. \quad (*) \end{aligned}$$

Since $(*)$ holds for any partition $\mathcal{P}$ with $|\mathcal{P}| < \delta$, we have

$$\text{length}(F_s) \geq \max_{1\leq j\leq n} \text{length}(f_s^j).$$

□

Example 3.2.11. Let $A = \mathrm{M}_{10}(\mathrm{C}[0,1])$. Define

$$u(t) = \begin{pmatrix} e^{-2\pi it\frac{9}{10}} & 0 & \cdots & 0 \\ 0 & e^{2\pi it\frac{1}{10}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi it\frac{1}{10}} \end{pmatrix}_{10\times 10}.$$

Then u is a unitary in A with $\det(u) = 1$ and $u \sim_h 1$.

Theorem 3.2.12. *Let $u \in M_{10}(C[0,1])$ be defined as in 3.2.11, then*

$$cel(u) \geq 2\pi \cdot \frac{9}{10}.$$

Proof. Let $\widetilde{F}_s$ be a path in $U(\mathrm{M}_{10}(\mathrm{C}[0,1]))$ with $\widetilde{F}_0 = 1 \in \mathrm{M}_{10}(\mathrm{C}[0,1])$ and $\widetilde{F}_1(t) = u(t)$. By Corollary 3.2.4, there is a path F_s in $U(\mathrm{M}_{10}(\mathrm{C}[0,1]))$ such that (1) $\|F - \widetilde{F}\| < \varepsilon/2$, (2) $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$, and (3) $|\mathrm{length}(\widetilde{F}) - \mathrm{length}(F)| < \varepsilon/2$. By Lemma 3.2.7 and Remark 3.2.8, there are continuous maps $f^1, f^2, ..., f^{10} : [0,1] \times [0,1] \to S^1$ and unitaries $U_s(t)$ such that $F_s(t) = U_s(t)\mathrm{diag}[f_s^1(t), f_s^2(t), ..., f_s^{10}(t)]U_s(t)^*$. By Lemma 3.2.10,

$$\mathrm{length}(F_s) \geq \max_{1 \leq i \leq 10}\{\mathrm{length}(f_s^i)\}.$$

Since $\|F - \widetilde{F}\| < \varepsilon/2$, $\|f_0^j - 1\| < \varepsilon/2$ for all $1 \leq j \leq 10$. For each fixed $t \in (\varepsilon, 1-\varepsilon)$, there exists one and only one j_0 such that

$$\|f_1^{j_0}(t) - e^{-2\pi it\frac{9}{10}}\| < \varepsilon/2.$$

For other $j \neq j_0$

$$\|f_1^j(t) - e^{2\pi it\frac{1}{10}}\| < \varepsilon/2.$$

(We use the fact that $\|e^{2\pi it\frac{1}{10}} - e^{-2\pi it\frac{9}{10}}\| \geq \varepsilon$ for $t \in (\varepsilon, 1-\varepsilon)$.) Since all f^j's are continuous, the index j_0 should be the same for all $t \in (\varepsilon, 1-\varepsilon)$.

Thus $f_s^{j_0}$ is a path in $U(\mathrm{C}[0,1])$ connecting a point near 1 and a point near $e^{-2\pi it\frac{9}{10}}$. By Lemma 3.1.4, $\mathrm{length}(f_s^{j_0}) \geq \frac{9}{10} \cdot 2\pi - \varepsilon$. Therefore,

$$\mathrm{length}(\tilde{F}_s) \geq \frac{9}{10} \cdot 2\pi - \varepsilon/2 - \varepsilon \geq \frac{9}{10} \cdot 2\pi - 2\varepsilon.$$

Since ε is arbitrary, we have $cel(u) \geq \frac{9}{10} \cdot 2\pi$, which completes the proof.

□

Example 3.2.13. Examples in some simple inductive limit C*-algebras.

Let $\{x_1, x_2, ...\}$ be a countable distinct dense subset of $[0,1]$ and let $\{k_n\}_{n=2}^{\infty}$ be a sequence of integers satisfying

$$\prod_n \frac{10^{k_n} - 1}{10^{k_n}} > \frac{11}{12}.$$

Let

$$A_1 = \mathrm{M}_{10}(\mathrm{C}[0,1]),\ A_2 = \mathrm{M}_{10^{k_2}}(A_1),\ ...,\ A_n = \mathrm{M}_{10^{k_n}}(A_{n-1}), ...$$

Define $\varphi_{n,n+1} : A_n \to A_{n+1}$ by

$$\varphi_{n,n+1}(f) = \begin{pmatrix} f & 0 & \cdots & 0 & 0 \\ 0 & f & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & f & 0 \\ 0 & 0 & \cdots & 0 & f(x_n) \end{pmatrix}_{10^{kn} \times 10^{kn}}$$

and $A = \varinjlim(A_i, \varphi_{i,i+1})$ be the inductive limit C*-algebra. Then A is simple.
Let $u(t) \in A_1$ be defined as in Example 3.2.11, then (see Theorem 3.2.15 and Corollary 3.2.16)

$$cel(\varphi_{1,\infty}(u)) \geq \frac{9}{10} \cdot 2\pi.$$

Inductive limit of such form was studied by Goodearl [27] and its exponential rank was calculated by Gong and Lin [26].

Lemma 3.2.14. *Let $\theta : P^L\mathbb{R} \to (\mathbb{R}^L, d_{max})$ be defined by*

$$\theta[x_1, x_2, \cdots, x_L] = (y_1, y_2, \cdots, y_L)$$

iff $[x_1, x_2, \cdots, x_L] = [y_1, y_2, \cdots, y_L]$ and $y_1 \leq y_2 \leq \cdots \leq y_L$. Then θ is an isometry.

Proof. Let $a = [a_1, a_2, \cdots, a_L]$, $b = [b_1, b_2, \cdots, b_L]$ be any two elements in $P^L\mathbb{R}$. Without loss of generality, we can assume that $a_1 \leq a_2 \leq \cdots \leq a_L$ and $b_1 \leq b_2 \leq \cdots \leq b_L$. Thus

$$d_{max}(\theta(a), \theta(b)) = \max_{1\leq i\leq L} |a_i - b_i|.$$

If $dist(a, b) \neq \max_{1\leq i\leq L} |a_i - b_i|$, then there exist a permutation $\sigma \in S_L$ such that

$$l \triangleq \max_{1\leq i\leq L} |a_i - b_{\sigma(i)}| < \max_{1\leq i\leq L} |a_i - b_i|.$$

Since $l < \max_{1\leq i\leq L} |a_i - b_i|$, there exists k such that $|a_k - b_k| > l$.
If $a_k < b_k$, then $|a_i - b_j| > l$ for any $i \leq k, j \geq k$. Since the cardinality of the set $\{\sigma(1), \sigma(2), \cdots, \sigma(k)\}$ is k, there is at least one element $i_0 \in \{1, 2, \cdots, k\}$ with $\sigma(i_0) \geq k$. Then $|a_{i_0} - b_{\sigma(i_0)}| > l$. Therefore,

$$\max_{1\leq i\leq k} |a_i - b_{\sigma(i)}| > l.$$

Similarly, if $a_k > b_k$, one can prove

$$\max_{k\leq i\leq L} |a_i - b_{\sigma(i)}| > l.$$

In either case, it contradicts $l = \max_{1\leq i\leq L} |a_i - b_{\sigma(i)}|$. Therefore,

$$dist(a, b) = \max_{1\leq i\leq L} |a_i - b_i| = d_{max}(\theta(a), \theta(b)),$$

which means θ is an isometry.

□

Theorem 3.2.15. *Let A_i, $\varphi_{i,i+1}$, $(i \in \mathbb{N})$ be defined as in 3.2.13, for any $\varepsilon \in (0, \frac{1}{100})$, let $u_\varepsilon \in A_1$ be defined by:*

$$u_\varepsilon(t) = \begin{pmatrix} e^{-2\pi it(\frac{9}{10}-\varepsilon)} & 0 & \cdots & 0 \\ 0 & e^{2\pi it(\frac{1}{10}-\varepsilon)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi it(\frac{1}{10}-\varepsilon)} \end{pmatrix}_{10\times 10}.$$

Then

$$cel(\varphi_{1,n}(u_\varepsilon)) \geq 2\pi(\frac{9}{10} - \varepsilon) - 5\varepsilon \quad \textit{for all } n \in \mathbb{N}.$$

Proof. By an easy calculation, we know that

$$\varphi_{1,n}(u_\varepsilon) \in M_L(C[0,1]), \quad \text{where } L = 10 \prod_{i=2}^{n} 10^{k_i}.$$

On the diagonal of $\varphi_{1,n}(u_\varepsilon)$, there are $\prod_{i=2}^{n}(10^{k_i} - 1)$ terms equal to $e^{-2\pi it(\frac{9}{10}-\varepsilon)}$, $9 \cdot \prod_{i=2}^{n}(10^{k_i} - 1)$ terms equal to $e^{2\pi it(\frac{1}{10}-\varepsilon)}$ and the rest are constants. Let α, β and γ denote, respectively, the numbers of terms of the forms $e^{-2\pi it(\frac{9}{10}-\varepsilon)}$, $e^{2\pi it(\frac{1}{10}-\varepsilon)}$ and constants on the diagonal of $\varphi_{1,n}(u_\varepsilon)$ (i.e. $\alpha = \prod_{i=2}^{n}(10^{k_i} - 1)$, $\beta = 9 \cdot \prod_{i=2}^{n}(10^{k_i} - 1)$ and $\gamma = L - \alpha - \beta$). Therefore $\frac{\alpha+\beta}{L} = \prod_{i=2}^{n} \frac{10^{k_i}-1}{10^{k_i}} > \frac{11}{12}$, which implies $\frac{\alpha}{\gamma} > \frac{11}{10}$.

For each $t \in [0,1]$, let $E(t)$ be the set consisting of all eigenvalues of $\varphi_{1,n}(u_\varepsilon)(t)$ (counting multiplicities). Define continuous functions $\bar{\bar{y}}_k(t)$ $(1 \leq k \leq L)$ from $[0,1]$ to $[-\frac{9}{10} + \varepsilon, \frac{1}{10} - \varepsilon]$ as follows:

$$\begin{aligned} \bar{\bar{y}}_k(t) &= -(\frac{9}{10} - \varepsilon)t, && \text{if } 1 \leq k \leq \alpha, \\ \bar{\bar{y}}_k(t) &= (\frac{1}{10} - \varepsilon)t, && \text{if } \alpha + 1 \leq k \leq \alpha + \beta, \\ \bar{\bar{y}}_k(t) &= -(\frac{9}{10} - \varepsilon)x_\bullet \text{ or } (\frac{1}{10} - \varepsilon)x_\bullet, && \text{if } \alpha + \beta + 1 \leq k \leq L, \end{aligned}$$

for $x_\bullet \in \{x_1, x_2, \cdots, x_n\}$ such that for all $t \in [0,1]$,

$$\{\exp\{2\pi i\bar{\bar{y}}_k(t)\} : 1 \leq k \leq L\} = E(t).$$

Let θ be the map defined in Lemma 3.2.14 and p_k $(1 \leq k \leq L)$ be projections from $\mathbb{R}^L$ to $\mathbb{R}$ with respect to the k-th coordinate. Define functions $\bar{y}_k(t) : [0,1] \to [-\frac{9}{10} + \varepsilon, \frac{1}{10} - \varepsilon]$ for $1 \leq k \leq L$ as follows:

$$\bar{y}_k(t) = p_k\theta[\bar{\bar{y}}_1(t), \bar{\bar{y}}_2(t), \cdots, \bar{\bar{y}}_L(t)].$$

Then $\bar{y}_k(t)$ $(1 \le k \le L)$ are continuous functions with $\bar{y}_1(t) \le \bar{y}_2(t) \le \cdots \le \bar{y}_L(t)$ for all $t \in [0,1]$ and

$$\{\bar{y}_1(t), \bar{y}_2(t), \cdots, \bar{y}_L(t)\} = \{\bar{\bar{y}}_1(t), \bar{\bar{y}}_2(t), \cdots, \bar{\bar{y}}_L(t)\}.$$

For each fixed $t \in (0,1)$, there are at most γ terms (the constants referred to above) in the set $\{\bar{y}_k(t)\}_{k=1}^L$ which could be less than $-(\frac{9}{10} - \varepsilon)t$. Therefore,

$$\bar{y}_k(t) \ge -(\frac{9}{10} - \varepsilon)t, \quad \text{for } k \ge \gamma + 1.$$

Similarly, for each fixed $t \in (0,1)$ there are at most $\gamma + \beta$ terms (the constants or terms of the form $(\frac{1}{10} - \varepsilon)t$) in the set $\{\bar{y}_k(t)\}_{k=1}^L$ which could be greater than $-(\frac{9}{10} - \varepsilon)t$. So

$$\bar{y}_k(t) \le -(\frac{9}{10} - \varepsilon)t, \quad \text{for } k \le L - \gamma - \beta = \alpha.$$

Since $\alpha > \gamma$,

$$\bar{y}_k(t) = -(\frac{9}{10} - \varepsilon)t, \quad \text{for } \gamma + 1 \le k \le \alpha.$$

Let $y_k(t) = \exp\{2\pi i \bar{y}_k(t)\}$ for $1 \le k \le L$. Then we have

$$y_k(t) = e^{-2\pi i t(\frac{9}{10} - \varepsilon)}, \quad \text{for } \gamma + 1 \le k \le \alpha,$$

and

$$\{y_k(t) : 1 \le k \le L\} = E(t), \quad \text{for all } t \in [0,1].$$

Let

$$W(t) = \begin{pmatrix} y_1(t) & 0 & \cdots & 0 \\ 0 & y_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_L(t) \end{pmatrix}.$$

Then for all $t \in [0,1]$, $W(t)$ and $\varphi_{1,n}(u_\varepsilon)(t)$ have exactly same eigenvalues (counting multiplicities). By Corollary 1.3 of [61], there exists $\Lambda(t) \in U(M_L(C[0,1]))$ such that

$$\|\Lambda(t)W(t)\Lambda(t)^* - \varphi_{1,n}(u_\varepsilon)(t)\| < \varepsilon,$$

for all $t \in [0,1]$. Therefore,

$$cel(\varphi_{1,n}(u_\varepsilon)) \ge cel(W) - \varepsilon\pi/2 > cel(W) - 2\varepsilon.$$

(Here we use two facts: for unitaries a, b, (1) $\|a - b\| < \varepsilon < 1$ implies $|cel(a) - cel(b)| < \varepsilon\pi/2$ and (2) $cel(uau^*) = cel(a)$ where u is a unitary.)

Let ε_j $(1 \le j \le L)$ be chosen such that $-\varepsilon < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_\alpha = 0 < \varepsilon_{\alpha+1} < \varepsilon_{\alpha+2} < \cdots < \varepsilon_L < \varepsilon$ and $\widetilde{y}_k(t) = \exp\{2\pi i(\bar{y}_k(t) + \varepsilon_k)\}$. Let

$$\widetilde{W}(t) = \begin{pmatrix} \widetilde{y}_1(t) & 0 & \cdots & 0 \\ 0 & \widetilde{y}_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{y}_L(t) \end{pmatrix} \in U(M_L(C[0,1])).$$

Then $\widetilde{W}(t)$ has no repeated eigenvalues for all $t \in [0,1]$ and

$$\|\widetilde{W}(t) - W(t)\| < \varepsilon.$$

In particular, $\widetilde{y}_\alpha(t) = e^{-2\pi i t(\frac{9}{10}-\varepsilon)}$.

Let $\widetilde{F}_s(t)$ be a rectifiable path in $U(M_L(C[0,1]))$ from 1 to $\widetilde{W}(t)$ with $\widetilde{F}_0(t) = 1$ and $\widetilde{F}_1(t) = \widetilde{W}(t)$. By Corollary 3.2.4, there is a smooth path F_s in $U(M_L(C[0,1]))$ such that $\|F - \widetilde{F}\| < \varepsilon$ and $F_1(t) = \widetilde{F}_1(t) = \widetilde{W}(t)$ and $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$ and $|\text{length}(\widetilde{F}) - \text{length}(F)| \leq \varepsilon$. By Lemma 3.2.7 and Remark 3.2.8, there exist continuous maps $f^1, f^2, ..., f^L : [0,1] \times [0,1] \longrightarrow S^1$ and unitaries $U_s(t)$ such that

$$F_s(t) = U_s(t)\text{diag}(f_s^1(t), f_s^2(t), ..., f_s^L(t))U_s(t)^*.$$

Since $F_1(t) = \widetilde{W}(t)$ and $\widetilde{y}_\alpha = e^{-2\pi i t(\frac{9}{10}-\varepsilon)}$, we can assume $f_1^\alpha = e^{-2\pi i t(\frac{9}{10}-\varepsilon)}$. Therefore, f_s^α is a path in $U(C[0,1])$ from a point near 1 to $e^{-2\pi i t(\frac{9}{10}-\varepsilon)}$. By Lemma 3.1.4,

$$\text{length}(f_s^\alpha) \geq 2\pi(\frac{9}{10} - \varepsilon) - \varepsilon.$$

Therefore,

$$\text{length}(\tilde{F}_s) \geq 2\pi(\frac{9}{10} - \varepsilon) - \varepsilon - \varepsilon = 2\pi(\frac{9}{10} - \varepsilon) - 2\varepsilon,$$

and

$$cel(\varphi_{1,n}(u_\varepsilon)) \geq 2\pi(\frac{9}{10} - \varepsilon) - 5\varepsilon.$$

□

Corollary 3.2.16. *Let* $A = \lim A_i$ *and* $u \in A_1$ *be defined as in 3.2.13. Then* $\varphi_{1,\infty}(u) \in CU(A)$ *and*

$$cel(\varphi_{1,\infty}(u)) \geq 2\pi \cdot \frac{9}{10}.$$

Note that for $A = \mathrm{M}_n(C[0,1])$ $(n \in \mathbb{N})$, $x \in CU(A)$ if and only if $\det(x(t)) = 1$ for each $t \in [0,1]$.

Proof. For any $\varepsilon \in (0, \frac{1}{100})$, let $u_\varepsilon \in A_1$ be defined as in Theorem 3.2.15, then

$$cel(\varphi_{1,n}(u_\varepsilon)) \geq 2\pi(\frac{9}{10} - \varepsilon) - 5\varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Since $\|\varphi_{1,n}(u) - \varphi_{1,n}(u_\varepsilon)\| \leq 2\pi\varepsilon$ for all $n \in \mathbb{N}$,

$$cel(\varphi_{1,n}(u)) \geq cel(\varphi_{1,n}(u_\varepsilon)) - 2\pi\varepsilon \geq 2\pi(\frac{9}{10} - 2\varepsilon) - 5\varepsilon.$$

Since ε is arbitrary, we have $cel(\varphi_{1,n}(u)) \geq 2\pi \cdot \frac{9}{10}$, for all $n \in \mathbb{N}$. Therefore,

$$cel(\varphi_{1,\infty}(u)) \geq 2\pi \cdot \frac{9}{10}.$$

□

Theorem 3.2.17. *Let A_i, $\varphi_{i,i+1}$, $(i \in \mathbb{N})$ be defined as in 3.2.13. For any $\varepsilon > 0$, there exists i such that $\frac{10^{k_i}-1}{10^{k_i}} \geq 1 - \frac{\varepsilon}{2\pi}$. Let $u \in A_i$ be defined by*

$$u(t) = \begin{pmatrix} e^{-2\pi i t \frac{10^{k_i}-1}{10^{k_i}}} & 0 & \cdots & 0 \\ 0 & e^{2\pi i t \frac{1}{10^{k_i}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i t \frac{1}{10^{k_i}}} \end{pmatrix}_{10^{k_i} \times 10^{k_i}} .$$

Then $\varphi_{i,\infty}(u) \in CU(A)$ and

$$cel(\varphi_{i,\infty}(u)) \geq 2\pi \frac{10^{k_i}-1}{10^{k_i}} \geq 2\pi(1 - \frac{\varepsilon}{2\pi}) = 2\pi - \varepsilon.$$

Proof. By repeating the way of the proof of Theorem 3.2.15 and Corollary 3.2.16, we can get the result.

□

Theorem 3.2.18. *Let A be defined as in 3.2.13. Then*

$$cel_{CU}(A) = 2\pi.$$

Proof. By Lemma 4.5 of [37], we know that $cel_{CU}(A) \leq 2\pi$. The equality holds by applying Theorem 3.2.17.

□

Bibliography

[1] B. Blackadar, *K-theory for operator Algebras*, Springer-Verlag, Heidelberg, 1986.

[2] R. Bhatia and C. Davis, *A bound for the spectral variation of a unitary operator*, Linear and Multi-linear Algebra 15(1984), 71-76.

[3] B. Blackadar and D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Anal. 45(1982), 297-340.

[4] M.-D.Choi and G.A. Elloitt, *Density of the selfadjoint elements with finite spectrum in an irrational rotation C^*-algebra*, Math. Scand., 67 (1990), 73-86.

[5] M. Dadarlat, *Reduction to dimension three of local spectra of real rank zero C^*-algebras*, J. Reine Angew. Math., 460 (1995), 189-212.

[6] M. Dadarlat and G. Gong, *A classification result for approximately homogeneous C^*-algebras of real rank zero*, GAFA, Geom. Funct. Anal., 7 (1997), 646-711.

[7] M. Dadarlat and T. Loring, *Classifying C^*-algebras via ordered, mod-p K-theory*, Math. Ann., 305 (1996), 601-616.

[8] M. Dadarlet, G. Nagy, A. Némethi and C. Pasnicu, *Reduction of topological stable rank in inductive limits of C^*-algebras*, Pacific J. Math., 153(2) (1992), 267-276.

[9] M. Dadarlet and A. Némethi, *shape theory and (connective) K-theory*, J. Operator Theory, 23 (1990), 207-291.

[10] S. Eilers, *A complete invariant for AD algebras with bounded dimension drop in K_1*, J. Funct. Anal., 139 (1996), 325-348.

[11] G. A. Elliott, *On the classification of inductive limits of semi-simple finite dimensional algebras*, J. Algebra, 38 (1976), 29-44.

[12] G. A. Elliott, *Dimension groups with torsion*, Internat. J. of Math., 1 (1990), 361-380.

[13] G. A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. 443 (1993), 179-219.

[14] G. A. Elliott, *A classification of certain simple C^*-algebras*, Quantum Noncommutative Analysis (editors H. Arakc et al.), Kluwer, Dordrecht, (1993), 373-385.

[15] G. A. Elliott, *A classification of certain simple C^*-algebras*, II, J. Ramanujan Math. Soc., 12 (1) (1997), 97-134.

[16] G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero*, II, Annals of Math., 144 (1996), 497-610.

[17] G. A Elliott and G. Gong, *On inductive limits of matrix algebras over the two-torus*, Amer. J. of Math., 118 (1996), 263-290.

[18] G. A. Elliott; G. Gong; L. Li, *On the classification of simple inductive limit C^*-algebras, II, the isomorphism theorem,* Invent. Math. 168 (2007), no. 2, 249-320.

[19] G. A. Elliott, G. Gong, H. Lin and C. Pasnicu, *Abelian C^*-subalgebras of C^*-algebras of real rank zero and inductive limit C^*-algebras*, Duke Math. J., 85 (1996), 511-554.

[20] G. A. Elliott, G. Gong, H. Lin and C. Pasnicu, *Homomorphisms, homotopies and approximations by circle algebras*, C. R. Math. Rep. Acad. Sci. Canada, XVI (1994), 45-50.

[21] G. A. Elliott, L. Robert and L. Santiago, *The cone of lower semicontinuous traces on a C^*-algebra*, American Journal of Mathematics, Vol 133, No. 4, Aug. 2011, 969-1005.

[22] G.Gong, *On inductive limits of matrix algebras over higher dimensional spaces*, Part II, Math. Scand., 80 (1997), 56-100.

[23] G.Gong, *On the classification of C^*-algebras of real rank zero and unsuspended E-equivalence types*, J. Funct. Anal., 152(2):281-329, 1998.

[24] G.Gong, *On the classification of simple inductive limits C^*-algebras, I: The Reduction Theorem*, Documenta Math. 7 (2002), 255-461.

[25] G.Gong, Chunlan Jiang, L. Li, C. Pasnicu, *A $\mathbb{T}$ structure for AH algebras with the ideal property and torsion free K-theory*, J.Funct.Anal.(2009), doi:10.1016/j.jfa.2009.11.016.

[26] G. Gong and H. Lin, *The exponential rank of inductive limit C^*-algebras*, Math. Scand., 71(1992), 301-319.

[27] K. R. Goodearl, *Notes on a class of simple C^*-algebras with real rank zero*, Publicacions Matemàtiques, Vol 36 (1992), 637-654.

[28] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1974.

[29] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, vol. 1, Elementary theory (New York: Academic Press, 1983).

[30] K. Ji and C.L. Jiang, *A complete classification of AI algebras with the ideal property*, Accepted by Canadian J. of Math.

[31] X.Jiang and H.Su, *A classification of Simple Limits of Splitting Interval Algebras*, Journal of Functional Analysis, no. 151. 50-76 (1997).

[32] C. Jiang and K. Wang, *A complete classification of limits of splitting interval algebras with the ideal property*, J. Ramanujan Math. Soc. 27, No.3 (2012) 305-354.

[33] E. Kirchberg and M. Rordam, *Infinite non-simple C*-algebras: absorbing the Cuntz algebra $\mathcal{O}_\infty$*, Adv. Math. 167(2002), 195-264.

[34] L. Li, *On the classification of simple C^*-algebras : Inductive limits of matrix algebras over tress*, Mem. Amer. Math. Soc., no. 605, vol. 127, 1997.

[35] L. Li, *Simple inductive limit C^*-algebras : Spectra and approximation by interval algebras*, J. Reine Angew. Math. 507 (1999), 57-79.

[36] H. Lin, *Exponential rank of C^*-algebras with real rank zero and Brown-Pedersen conjectures*, J. Funct. Anal., 114(1993) no. 1, 1-11.

[37] H. Lin, *Exponentials in simple $\mathcal{Z}$-stable C^*-algebras*, J. Funct. Anal., 266 (2014), no. 2, 754-791.

[38] H. Lin, *An Introduction to the classification of amenable C^*-algebras*, World Scientific, 2001.

[39] H. Lin, *Generalized Weyl-von Neumann Theorems*, Int. J. Math., 02, 725 (1991).

[40] H. Lin, *Generalized Weyl-von Neumann Theorems II*, Math. Scand. 77 (1995), 129-147.

[41] H. Lin, *Simple nuclear C^*-algebras of tracial topological rank one*, J. Funct. Anal., 251 (2007), no. 2, 601-679.

[42] J. R. Munkres, *Analysis on manifolds*, Westview Press, Jul 7, 1997.

[43] C. Pasnicu, *On inductive limits of certain C^*-algebras of the form $C(X)\bigotimes F$*, Trans. Amer. Math. Soc., 310 (1998), 703-714.

[44] C. Pasnicu, *Shape equivalence, non-stable K-theory and AH algebras*, Pacific J. Math. 192 (2000), 159-182.

[45] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press INC. (LONDON) LTD.

[46] N. C. Phillips, *Simple C^*-algebras with the property weak (FU)*, Math. Scand. 69(1991), 127-151.

[47] N. C. Phillips, *The rectifiable metric on the space of projections in a C^*-algebra*, Internat. J. Math. 3(1992), 679-698.

[48] N. C. Phillips, *How many exponentials?*, Amer. J. Math., Vol. 116, No. 6 (Dec., 1994), 1513-1543.

[49] N. C. Phillips, *The C^* projective length of n-homogeneous C^*-algebras*, J. Operator Theory, 31(1994), 253-276.

[50] N. C. Phillips, *Reduction of exponential rank in direct limits of C^*-algebras*, Can. J. Math. Vol. 46(4), 1994, 818-853.

[51] N. C. Phillips, *Approximation by unitaries with finite spectrum in purely infinite C^*-algebras*, J. Funct. Anal., 120 (1994), 98-106.

[52] N. C. Phillips, *Exponential length and traces*, Proc. Roy. Soc. Edinburgh Sect. A, Vol. 125, January 1995, 13-29.

[53] N. C. Phillips and J. R. Ringrose, *Exponential rank in operator algebras, Current topics in operator algebras*, (Nara, 1990), 395-413, World Sci. Publ., River Edge, NJ, 1991.

[54] F. Perera and A. S. Toms, *Recasting the Elliott Conjecture*, Math. Ann. 338, (2007), pp. 669-702.

[55] J. R. Ringrose, *Exponential length and exponential rank in C^*-algebras*, Proc. London Math. Soc. (3) 46(1983), 301-333.

[56] M. Rørdam, *On the structure of simple C^*-algebras tensored with a* UHF*-algebra. II.* J. Funct. Anal. 107(1992), 255-269.

[57] M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C^*-algebras*, Int. J. Math. 15(2004), 1065-1084.

[58] M. Rørdam, F. Larsen and N. J. Laustsen, *An introduction to K-theory for C^*-algebras.* Cambridge University press, 2000.

[59] K. H. Stevens, *The classification of certain non-simple approximate interval algebras*, Fields Institute Communications 20 (1998), 105-148.

[60] H. Su, *On the classification of C^*-algebras of real rank zero: Inductive limits of matrix algebras over non-Hausdorff graph*, Mem. Amer. Math. Soc. No 547, vol. 114 (1995).

[61] K. Thomsen, *Homomorphisms between finite direct sums of circle algebra*, Linear and Multi-linear Algebra 1992, Vol.32, 33-50.

[62] K. Thomsen, *On the reduced C^*-exponential length*, Operator algebras and quantum field theory (Rome, 1996), 59-64, Int. Press, Cambridge, MA, 1997.

[63] K. Thomsen, *Traces, Unitary Characters and Crossed Products by* $\mathbb{Z}$, Publ. RIMS, Kyoto Univ. 31 (1995), 1011-1029.

[64] N. E. Wegge-Olsen, *K-Theory and C^*-algebras*, Oxford University press, 1993.

[65] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller differentialgleichungen*, Math. Ann., 71 (1912), 441-479.

[66] S. Zhang, *On the exponential rank and exponential length of C^*-algebras*, J. Operator Theory, 28 (1992), 337-355.

[67] S. Zhang, *Exponential rank and exponential length of operators on Hilbert C^*-algebras*, Ann. of Math. 137 (1993), 121-144.

[68] S. Zhang, *Factorizations of invertible operators and K-theory of C^*-algebras*, Bull. Amer. Math. Soc., 28(1993), 75-83.